Matematika 2

Dvojni in trojni integrali

(1) Izračunaj integrale:

(a) $\int_0^2 dy \int_0^1 (x^2 + 2y)\, dx$,

(b) $\int_0^1 dx \int_x^1 (x + y)\, dy$,

(c) $\int_1^2 dx \int_{1/x}^x \frac{x^2}{y^2}\, dy$.

Rešitev:

(a)
$$\begin{aligned}\int_0^2 dy \int_0^1 (x^2 + 2y)\, dx &= \int_0^2 dy \left(\frac{x^3}{3} + 2xy\right)\Bigg|_{x=0}^{x=1} = \int_0^2 dy \left(\frac{1}{3} + 2y\right),\\ &= \left(\frac{y}{3} + y^2\right)\Bigg|_{y=0}^{y=2} = \frac{2}{3} + 4,\\ &= \frac{14}{3}.\end{aligned}$$

(b)
$$\begin{aligned}\int_0^1 dx \int_x^1 (x + y)\, dy &= \int_0^1 dx \left(xy + \frac{y^2}{2}\right)\Bigg|_{y=x}^{y=1} = \int_0^1 dx \left(\left(x + \frac{1}{2}\right) - \left(x^2 + \frac{x^2}{2}\right)\right),\\ &= \int_0^1 dx \left(-\frac{3x^2}{2} + x + \frac{1}{2}\right) = \left(-\frac{x^3}{2} + \frac{x^2}{2} + \frac{x}{2}\right)\Bigg|_{x=0}^{x=1},\\ &= -\frac{1}{2} + \frac{1}{2} + \frac{1}{2},\\ &= \frac{1}{2}.\end{aligned}$$

(c)
$$\begin{aligned}\int_1^2 dx \int_{1/x}^x \frac{x^2}{y^2}\, dy &= \int_1^2 dx \left(-\frac{x^2}{y}\right)\Bigg|_{y=\frac{1}{x}}^{y=x} = \int_1^2 dx \left(-x + x^3\right),\\ &= \left(-\frac{x^2}{2} + \frac{x^4}{4}\right)\Bigg|_{x=1}^{x=2} = (-2 + 4) - \left(-\frac{1}{2} + \frac{1}{4}\right),\\ &= \frac{9}{4}.\end{aligned}$$

□

(2) Izračunaj integrale:

(a) funkcije $f(x, y) = x^2 + xy^3$ po pravokotniku $0 \le x \le 1$, $1 \le y \le 2$,

(b) funkcije $f(x, y) = xy$ po trikotniku, ki ga omejujejo premice $y = 0$, $y = x$ in $x = 1$,

(c) funkcije $f(x, y) = x^2 y$ po liku med krivuljama $y = x$ in $y = x^2$.

Rešitev: (a) Integrirali bomo po območju $x \in [0,1]$ in $y \in [1,2]$, kar nam da:

$$\begin{aligned}
\int_0^1 dx \int_1^2 (x^2 + xy^3)\, dy &= \int_0^1 dx \left(x^2 y + \frac{xy^4}{4} \right) \Bigg|_{y=1}^{y=2} = \int_0^1 dx \left((2x^2 + 4x) - \left(x^2 + \frac{x}{4} \right) \right), \\
&= \int_0^1 dx \left(x^2 + \frac{15x}{4} \right) = \left(\frac{x^3}{3} + \frac{15x^2}{8} \right) \Bigg|_{x=0}^{x=1} = \frac{1}{3} + \frac{15}{8}, \\
&= \frac{53}{24}.
\end{aligned}$$

(b) Trikotnik, ki ga omejujejo premice $y = 0$, $y = x$ in $x = 1$, parametriziramo z $x \in [0,1]$ in $y \in [0,x]$. Sledi

$$\int_0^1 dx \int_0^x xy\, dy = \int_0^1 dx \left(\frac{xy^2}{2} \right) \Bigg|_{y=0}^{y=x} = \int_0^1 dx \left(\frac{x^3}{2} \right) = \frac{x^4}{8} \Bigg|_{x=0}^{x=1} = \frac{1}{8}.$$

(c) Območje med parabolo in premico lahko parametriziramo z $x \in [0,1]$ in $y \in [x^2, x]$, kar nam da:

$$\begin{aligned}
\int_0^1 dx \int_{x^2}^x x^2 y\, dy &= \int_0^1 dx \left(\frac{x^2 y^2}{2} \right) \Bigg|_{y=x^2}^{y=x} = \int_0^1 dx \left(\frac{x^4}{2} - \frac{x^6}{2} \right) = \left(\frac{x^5}{10} - \frac{x^7}{14} \right) \Bigg|_{x=0}^{x=1}, \\
&= \frac{1}{10} - \frac{1}{14} = \frac{1}{35}.
\end{aligned}$$

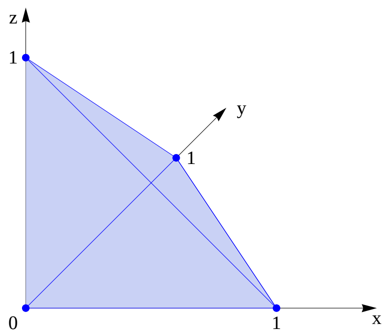
□

(3) Izračunaj volumen telesa, ki leži pod grafom funkcije $f(x,y) = 1 - x - y$ nad trikotnikom, ki ga omejujejo premice $x = 0$, $y = 0$ in $y = 1 - x$.

Rešitev: Volumen telesa, ki leži pod grafom funkcije dveh spremenljivk f nad likom L, je enak

$$V = \iint_L f(x,y)\, dx\, dy.$$

Naše telo ima obliko štiristrane piramide.



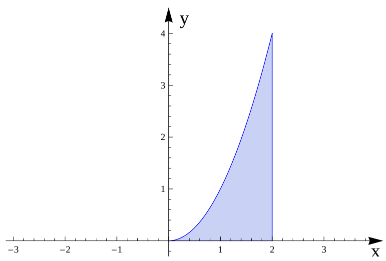
Njen volumen je enak:

$$
\begin{aligned}
V = \int_0^1 dx \int_0^{1-x} (1-x-y)\,dy &= \int_0^1 dx\, \left(y - xy - \frac{y^2}{2}\right)\Bigg|_{y=0}^{y=1-x}, \\
&= \int_0^1 dx\, \left((1-x) - x(1-x) - \frac{(1-x)^2}{2}\right), \\
&= \int_0^1 dx\, \left(1 - x - x + x^2 - \frac{1-2x+x^2}{2}\right), \\
&= \int_0^1 dx\, \left(\frac{x^2}{2} - x + \frac{1}{2}\right), \\
&= \left(\frac{x^3}{6} - \frac{x^2}{2} + \frac{x}{2}\right)\Bigg|_{x=0}^{x=1} = \frac{1}{6} - \frac{1}{2} + \frac{1}{2}, \\
&= \frac{1}{6}.
\end{aligned}
$$

□

(4) Izračunaj središče lika, ki je omejen s krivuljami $y = x^2$, $y = 0$ in $x = 2$.

Rešitev: Iščemo središče lika, ki ima za zgornji rob kvadratno parabolo.



Središče ravninskega lika L lahko izračunamo s pomočjo formul:

$$
x_* = \frac{\iint_L x\,dx\,dy}{S}, \qquad y_* = \frac{\iint_L y\,dx\,dy}{S},
$$

kjer je S ploščina lika L, ki je v našem primeru enaka

$$
S = \int_0^2 x^2\,dx = \frac{x^3}{3}\Bigg|_0^2 = \frac{8}{3}.
$$

Sedaj bomo izračunali oba integrala v števcih zgornjih ulomkov. Naš lik lahko parametriziramo s predpisoma $x \in [0,2]$ in $y \in [0, x^2]$, od koder dobimo:

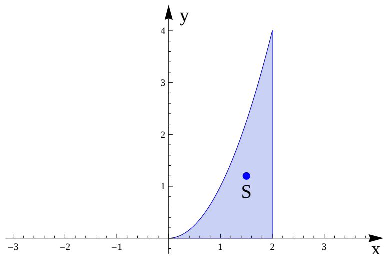
$$
\iint_L x\,dx\,dy = \int_0^2 dx \int_0^{x^2} x\,dy = \int_0^2 dx\,(xy)\Big|_{y=0}^{y=x^2} = \int_0^2 x^3\,dx = \frac{x^4}{4}\Bigg|_{x=0}^{x=2} = 4,
$$

$$
\iint_L y\,dx\,dy = \int_0^2 dx \int_0^{x^2} y\,dy = \int_0^2 dx\,\left(\frac{y^2}{2}\right)\Bigg|_{y=0}^{y=x^2} = \int_0^2 \frac{x^4}{2}\,dx = \frac{x^5}{10}\Bigg|_{x=0}^{x=2} = \frac{32}{10}.
$$

Koordinati središča lika sta torej:

$$x_* = \frac{\iint_L x\,dx\,dy}{S} = \frac{4}{\frac{8}{3}} = \frac{3}{2},$$
$$y_* = \frac{\iint_L y\,dx\,dy}{S} = \frac{\frac{32}{10}}{\frac{8}{3}} = \frac{6}{5}.$$

Poglejmo še skico.



□

(5) S prevedbo na polarne koordinate izračunaj integrala:

(a) $\displaystyle\iint_D \frac{dx\,dy}{1-x^2-y^2}$, kjer je D krog $x^2+y^2 \leq 1/4$,

(b) $\displaystyle\iint_D \sqrt{x^2+y^2}\,dx\,dy$, kjer je D kolobar $4 \leq x^2+y^2 \leq 9$.

Rešitev: Kadar integriramo po krogih ali kolobarjih, je namesto v kartezičnih lažje računati v polarnih koordinatah. Pri tem uporabljamo substitucijo:

$$x = r\cos\phi,$$
$$y = r\sin\phi.$$

Izraz $dx\,dy$ moramo v tem primeru zamenjati z izrazom $r\,dr\,d\phi$. Velja si zapomniti zvezi:

$$x^2+y^2 = r^2,$$
$$\sqrt{x^2+y^2} = r.$$

(a) Integrirali bomo funkcijo $f(x,y) = \frac{1}{1-x^2-y^2} = \frac{1}{1-r^2}$ po krogu s polmerom $R = \frac{1}{2}$. V polarnih koordinatah ga parametriziramo s predpisoma $\phi \in [0, 2\pi]$ in $r \in \left[0, \frac{1}{2}\right]$. Sledi

$$\iint_D \frac{dx\,dy}{1-x^2-y^2} = \int_0^{2\pi} d\phi \int_0^{\frac{1}{2}} \frac{1}{1-r^2} r\,dr.$$

Uvedimo novo spremenljivko $1-r^2 = t$, ki nam da $-2r\,dr = dt$ in

$$\int \frac{1}{1-r^2} r\,dr = -\frac{1}{2}\int \frac{dt}{t} = -\frac{1}{2}\ln t + C = -\frac{1}{2}\ln(1-r^2) + C.$$

Od tod dobimo:

$$\begin{aligned}\iint_D \frac{dx\,dy}{1-x^2-y^2} &= \int_0^{2\pi} d\phi \int_0^{\frac{1}{2}} \frac{1}{1-r^2} r\,dr = \int_0^{2\pi} d\phi \left(-\frac{1}{2}\ln(1-r^2)\right)\Bigg|_{r=0}^{r=\frac{1}{2}},\\ &= \int_0^{2\pi} d\phi \left(-\frac{1}{2}\ln\left(1-\frac{1}{4}\right)\right) = -\frac{1}{2}\ln\left(\frac{3}{4}\right)\int_0^{2\pi} d\phi = -\frac{1}{2}\ln\left(\frac{3}{4}\right)\cdot \phi\Big|_{\phi=0}^{\phi=2\pi},\\ &= -\pi\ln\left(\frac{3}{4}\right).\end{aligned}$$

(b) V tem primeru integriramo po kolobarju $4 \leq x^2+y^2 \leq 9$, ki ga lahko parametriziramo s predpisoma $\phi \in [0, 2\pi]$ in $r \in [2, 3]$. Torej je

$$
\begin{aligned}
\iint_D \sqrt{x^2+y^2}\,dx\,dy &= \int_0^{2\pi} d\phi \int_2^3 r\,r\,dr = \int_0^{2\pi} d\phi \left(\frac{r^3}{3}\right)\Bigg|_{r=2}^{r=3}, \\
&= \int_0^{2\pi} d\phi \left(\frac{27}{3} - \frac{8}{3}\right) = \frac{19}{3}\int_0^{2\pi} d\phi = \frac{19}{3}\phi\Big|_{\phi=0}^{\phi=2\pi}, \\
&= \frac{38\pi}{3}.
\end{aligned}
$$

□

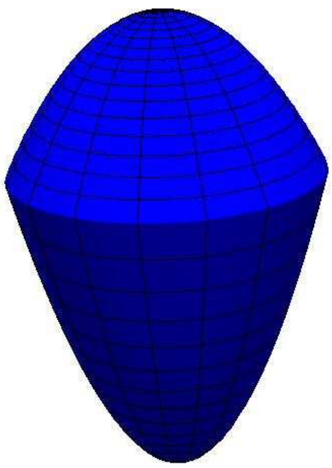
(6) Izračunaj volumen telesa, ki ga omejujeta ploskvi $z = 3 - x^2 - y^2$ in $z = 2x^2 + 2y^2$.

Rešitev: Najprej izračunajmo presečišče danih ploskev. Veljati mora:

$$
\begin{aligned}
2x^2 + 2y^2 &= 3 - x^2 - y^2, \\
x^2 + y^2 &= 1.
\end{aligned}
$$

Tloris telesa je torej omejen s krožnico $x^2 + y^2 = 1$. Zgornja ploskev se ujema z grafom funkcije $f(x, y) = 3 - x^2 - y^2$, spodnja pa z grafom funkcije $g(x, y) = 2x^2 + 2y^2$.

Dobljeno telo ima naslednjo obliko.



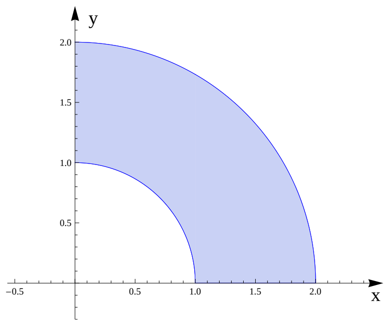
Ker integriramo po krogu s polmerom $R = 1$, bomo raje računali v polarnih koordinatah. Iz enakosti $3 - x^2 - y^2 = 3 - r^2$ in $2x^2 + 2y^2 = 2r^2$ dobimo:

$$
\begin{aligned}
V = \iint_K (f(x,y) - g(x,y))\,dx\,dy &= \int_0^{2\pi} d\phi \int_0^1 (3 - r^2 - 2r^2) r\,dr, \\
&= \int_0^{2\pi} d\phi \int_0^1 (3r - 3r^3)\,dr = \int_0^{2\pi} d\phi \left(\frac{3r^2}{2} - \frac{3r^4}{4}\right)\Bigg|_{r=0}^{r=1}, \\
&= \int_0^{2\pi} d\phi \left(\frac{3}{2} - \frac{3}{4}\right) = \frac{3}{4}\int_0^{2\pi} d\phi, \\
&= \frac{3\pi}{2}.
\end{aligned}
$$

□

(7) Izračunaj središče kolobarja $1 \leq x^2 + y^2 \leq 4$, $x \geq 0$, $y \geq 0$.

Rešitev: Naš lik je del kolobarja, ki leži v prvem kvadrantu.



Lik bomo parametrizirali v polarnih koordinatah s predpisoma $\phi \in \left[0, \frac{\pi}{2}\right]$ in $r \in [1, 2]$. Njegova ploščina je enaka četrtini razlike ploščin večjega in manjšega kroga, kar je enako $S = \frac{1}{4}\left(\pi \cdot 2^2 - \pi \cdot 1^2\right) = \frac{3\pi}{4}$. Poleg tega je:

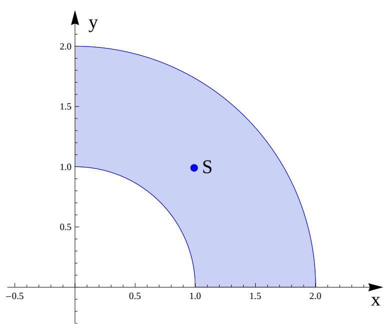
$$\begin{aligned}\iint_L x\,dx\,dy &= \int_0^{\frac{\pi}{2}} d\phi \int_1^2 r\cos\phi\, r\,dr = \int_0^{\frac{\pi}{2}} d\phi \left(\cos\phi\,\frac{r^3}{3}\right)\Bigg|_{r=1}^{r=2}, \\ &= \int_0^{\frac{\pi}{2}} d\phi \left(\frac{7}{3}\cos\phi\right) = \frac{7}{3}\int_0^{\frac{\pi}{2}} \cos\phi\,d\phi = \frac{7}{3}\sin\phi\Bigg|_{\phi=0}^{\phi=\frac{\pi}{2}} = \frac{7}{3},\end{aligned}$$

$$\begin{aligned}\iint_L y\,dx\,dy &= \int_0^{\frac{\pi}{2}} d\phi \int_1^2 r\sin\phi\, r\,dr = \int_0^{\frac{\pi}{2}} d\phi \left(\sin\phi\,\frac{r^3}{3}\right)\Bigg|_{r=1}^{r=2}, \\ &= \int_0^{\frac{\pi}{2}} d\phi \left(\frac{7}{3}\sin\phi\right) = \frac{7}{3}\int_0^{\frac{\pi}{2}} \sin\phi\,d\phi = -\frac{7}{3}\cos\phi\Bigg|_{\phi=0}^{\phi=\frac{\pi}{2}} = \frac{7}{3}.\end{aligned}$$

Od tod dobimo koordinati središča lika:

$$\begin{aligned}x_* &= \frac{\iint_L x\,dx\,dy}{S} = \frac{\frac{7}{3}}{\frac{3\pi}{4}} = \frac{28}{9\pi}, \\ y_* &= \frac{\iint_L y\,dx\,dy}{S} = \frac{\frac{7}{3}}{\frac{3\pi}{4}} = \frac{28}{9\pi}.\end{aligned}$$

Poglejmo še skico.



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