Theoretical Astrophysics
Homework #1
18. november 2021

Please, upload the results of the homework on the online classroom. Deadline: three weeks!

Exercise 1: The equations of the stellar structure, in general, cannot be solved analytically, and we must resort to numerical modelling. However, the mechanical part of the stellar structure equations can be solved separately from energy equations if there is a relationship between stellar pressure and density, that is simple enough.

Assuming a simple relationship between pressure and density $p = K\rho^{\gamma}$, we can derive the so-called Lane-Emden equation:

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta^n, \tag{1}$$

where $\rho(r) \equiv \rho_c\theta^n(r)$, $\gamma = \frac{n+1}{n}$, $r \equiv \alpha\xi$ and

$$\alpha = \left[(n+1)\frac{K{\rho_c}^{\frac{1-n}{n}}}{4\pi G}\right]^{1/2}. \tag{2}$$

This stellar model is called a *polytropic* model.

a) During the exercises in class, we wrote the boundary conditions and derived Lane-Emden's analytical solution in the two cases, $n = 0$ and $n = 1$. You have now to check if the solution in the case $n = 5$ is

$$\theta = \frac{1}{\sqrt{1 + \frac{1}{3}\xi^2}} \tag{3}$$

(write the derivation).
b) Write a code (in the language you prefer) to numerically find the solution to eq. (1) for any polytropic index n. Write how you structured your code. Compute the first zero of the solution ξ_1, the derivative $\frac{d\theta}{d\xi}\mid_{\xi_1}$ and $M_n = -\xi_1^2\left(\frac{d\theta}{d\xi}\right)\mid_{\xi_1}$. Show your numerical results for the n indexes in the following table:

n	ξ_1	$\frac{d\theta}{d\xi}\mid_{\xi_1}$	M_n
0.0			
0.1			
0.5			
1.0			
1.5			
3.0			
5.0			

c) For models with indexes $n = 1.5$ (monoatomic gas) and $n = 3$ (relativistic degenerate gas) draw how density (ρ/ρ_c), mass (m/M), pressure (p/p_c) and temperature (T/T_c) behave as a function of radius (r/R, where $R = \alpha\xi_1$) and compare the results for both models (describe and explain the graphs). To calculate the temperature assume that there is no radiation pressure and that the equation of state for the ideal gas applies.

č) Compare your results for the model with $n = 3$ (in physical units) with the results for the standard solar model. The values for the latter model can be found in a table at http://www.sns.ias.edu/ jnb/SNdata/Export/BS2005/bs05_agsop.dat. Details regarding the model are written in the paper on that website. Describe the graphs and explain why and where do the two models differ.

d) Derive the equation for the gravitational energy of the star described by a polytropic model with a general index n. Show that the following relation holds:

$$d\left(\frac{p}{\rho}\right) = \frac{1}{1+n}\left(\frac{dp}{\rho}\right) ,$$

and write the general equation for the gravitational energy and then use the *per partes* integration.

e) Use the result of the previous task and write down the total energy of the star for a general case. What is the total energy of the star for the case when $n = 3$?