Kolokvij 7. 5. 2021

1. naloga (20 točk)

Dana je funkcija f s predpisom $f(x) = (x+1)e^{-x}$.

a) Dokaži, da je f skrčitev na intervalu $[0,1]$ in da ima enačba $f(x) = x$ natanko eno pozitivno rešitev.

b) Pozitivno rešitev enačbe $f(x) = x$ določi na tri decimalke natančno kot limito zaporedja z rekurzivno formulo $x_{n+1} = f(x_n)$ in ustreznim začetnim členom.

a) $f'(x) = e^{-x} - (x+1)e^{-x} = -x e^{-x}$

$\Rightarrow$ f padajoča na $[0,\infty)$

$f(0) = 1$, $f(1) = 2e^{-1}$ $\Rightarrow$ $f([0,1]) = [2e^{-1}, 1] \subset [0,1]$

$f''(x) = -e^{-x} + x e^{-x} = (x-1)e^{-x} \leq 0$ za $x \in [0,1]$

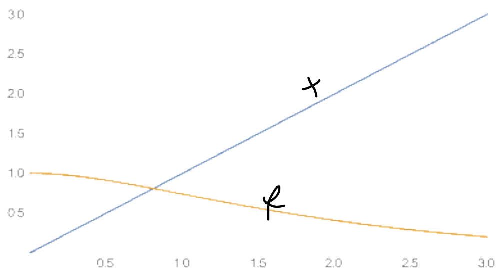
$\Rightarrow \max\limits_{x\in[0,1]} f'(x) = f'(0) = 0$ in $\min\limits_{x\in[0,1]} f'(x) = f'(1) = -e^{-1}$

$\Rightarrow q := \max\limits_{x\in[0,1]} |f'(x)| = e^{-1}$

$0 < q < 1 \Rightarrow$ f je skrčitev na $[0,1]$

Po Banachovem skrčitvenem načelu ima enačba $f(x) = x$ natanko eno rešitev x_0 na $[0,1]$. Ker je $f(0) = 1 \neq 0$, je $x_0 > 0$.

Ker je f padajoča, x pa naraščajoča na $[0,\infty)$, je x_0 edina pozitivna rešitev enačbe $f(x) = x$



b) $\frac{q}{1-q} \approx 0.58$

$|x_n - x_0| \leq \frac{q}{1-q} |x_n - x_{n-1}|$

$\Rightarrow$ Če se x_n in x_{n-1} ujemata v treh decimalkah, je x_n približek x_0 na tri decimalke.

Za x_1 lahko izberemo katerokoli število iz $[0,1]$, npr. $x_1 = 1$.

$x_2 = f(x_1) \approx 0.735759$

$x_3 = f(x_2) \approx 0.831674$

$x_4 = f(x_3) \approx 0.797364$

$x_5 = f(x_4) \approx 0.809739$
$x_6 = f(x_5) \approx 0.805287$
$x_7 = f(x_6) \approx 0.80689$
$x_8 = f(x_7) \approx 0.806313$
$\Rightarrow x_0 \approx 0.806$

2. naloga (25 točk)

Funkcijo

$$f(x) = \begin{cases} -x; & -\pi \le x < 0 \\ 3x; & 0 \le x \le \pi \end{cases}$$

a) Funkcijo f razvij v Fourierovo vrsto na intervalu $[-\pi, \pi]$.

b) Nariši graf Fourierove vrste kot funkcije na $\mathbb{R}$.

c) S pomočjo Fourierovega razvoja funkcije f izračunaj $\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2}$.

a) $\tilde{f}(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx))$

$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\left(-\int_{-\pi}^{0} x\,dx + 3\int_0^{\pi} x\,dx\right)$

x liha $\Rightarrow \int_{-\pi}^{0} x\,dx = -\int_0^{\pi} x\,dx$

$\Rightarrow a_0 = \frac{4}{\pi}\int_0^{\pi} x\,dx = \frac{2}{\pi} x^2 \Big|_0^{\pi} = 2\pi$

$n \in \mathbb{N}$: $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx = \frac{1}{\pi}\left(-\int_{-\pi}^{0} \underbrace{x\cos(nx)}_{\text{liha}}\,dx + 3\int_0^{\pi} x\cos(nx)\,dx\right) =$

$= \frac{4}{\pi}\int_0^{\pi} \underset{u}{x}\,\underbrace{\cos(nx)\,dx}_{dv} = \frac{4}{\pi}\left(\underbrace{\frac{x}{n}\sin(nx)\Big|_0^{\pi}}_{0} - \frac{1}{n}\int_0^{\pi} \sin(nx)\,dx\right) =$

$du = dx \quad v = \frac{1}{n}\sin(nx)$

$= \frac{4}{\pi}\cdot\frac{1}{n^2}\cos(nx)\Big|_0^{\pi} = \frac{4}{\pi n^2}\left((-1)^n - 1\right) = \begin{cases} 0; & n \text{ sod} \\ -\frac{8}{\pi n^2}; & n \text{ lih} \end{cases}$

$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx = \frac{1}{\pi}\left(-\int_{-\pi}^{0} x\sin(nx)\,dx + 3\int_0^{\pi} x\sin(nx)\,dx\right)$

$x\sin(nx)$ soda $\Rightarrow \int_{-\pi}^{0} x\sin(nx)\,dx = \int_0^{\pi} x\sin(nx)\,dx$

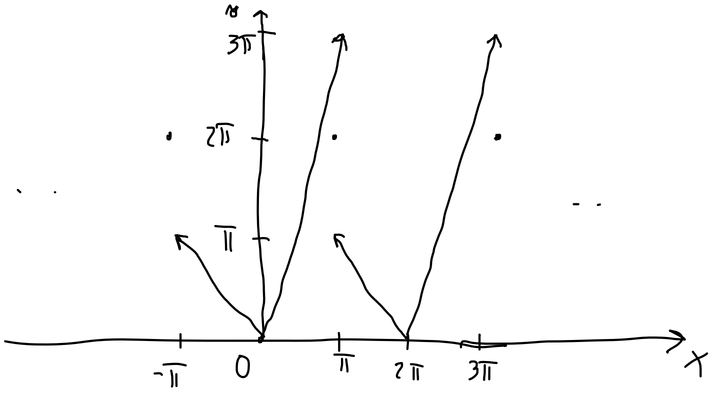
$\Rightarrow b_n = \frac{2}{\pi}\int_0^{\pi} \underset{u}{x}\,\underbrace{\sin(nx)\,dx}_{dv} = \frac{2}{\pi}\left(-\frac{x}{n}\cos(nx)\Big|_0^{\pi} + \frac{1}{n}\int_0^{\pi}\cos(nx)\,dx\right) =$

$du = dx \quad v = -\frac{1}{n}\cos(nx)$

$= \frac{2}{\pi}\left(-\frac{\pi}{n}(-1)^n + \underbrace{\frac{1}{n^2}\sin(nx)\Big|_0^{\pi}}_{0}\right) = \frac{2(-1)^{n+1}}{n}$

$\Rightarrow \tilde{f}(x) = \pi - \frac{8}{\pi}\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2}\cos((2k-1)x) + 2\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}\sin(nx)$

b)



c) $s := \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2}$

$\tilde{f}$ zv. v 0 $\Rightarrow \pi - \frac{8}{\pi} s = \tilde{f}(0) = f(0) = 0$

$\Rightarrow s = \frac{\pi^2}{8}$

3. naloga (20 točk)

Ploskev P je dana z enačbo $z = 1 - \frac{x^2}{4} - y^2$. V točki $(2, 0, 0)$ poišči glavni ukrivljenosti in glavni smeri ter Gaussovo in povprečno ukrivljenost ploskve P.

PARAMETRIZACIJA $\vec{r}(x,y) = (x, y, 1 - \frac{x^2}{4} - y^2)$

$(2,0,0) = \vec{r}(2,0)$

$\vec{r}_x = (1, 0, -\frac{x}{2})$ $\quad\quad$ $\vec{r}_y = (0, 1, -2y)$

$\vec{r}_x(2,0) = (1,0,-1)$ $\quad\quad$ $\vec{r}_y(2,0) = (0,1,0)$

$E = (1,0,-1)\cdot(1,0,-1) = 2$

$F = (1,0,-1)\cdot(0,1,0) = 0$

$G = (0,1,0)\cdot(0,1,0) = 1$

$(\vec{r}_x \times \vec{r}_y)(2,0) = (1,0,-1) \times (0,1,0) = (1,0,1)$

$\vec{n} = (\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}})$

$\vec{r}_{xx} = (0, 0, -\frac{1}{2})$ $\quad\quad$ $\vec{r}_{xy} = (0,0,0)$ $\quad\quad$ $\vec{r}_{yy} = (0,0,-2)$

$L = \vec{n}\cdot\vec{r}_{xx} = -\frac{1}{2\sqrt{2}}$ $\quad\quad$ $M = \vec{n}\cdot\vec{r}_{xy} = 0$ $\quad\quad$ $N = \vec{n}\,\vec{r}_{yy} = -\sqrt{2}$

$$A = \begin{bmatrix} E & F \\ F & G \end{bmatrix}^{-1} \begin{bmatrix} L & M \\ M & N \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} -\frac{1}{2\sqrt{2}} & 0 \\ 0 & -\sqrt{2} \end{bmatrix} =$$

$$= \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} -\frac{1}{2\sqrt{2}} & 0 \\ 0 & -\sqrt{2} \end{bmatrix} = \begin{bmatrix} -\frac{1}{4\sqrt{2}} & 0 \\ 0 & -\sqrt{2} \end{bmatrix}$$

$\Rightarrow$ Glavni ukrivljenosti sta $-\frac{1}{4\sqrt{2}}$ in $-\sqrt{2}$, pripadajoči glavni smeri pa $\vec{F}_x(2,0)=(1,0,-1)$ in $\vec{F}_y(2,0)=(0,1,0)$.

$\Rightarrow$ Gaussova ukrivljenost je $\frac{1}{4}$, povprečna ukrivljenost pa $\dfrac{-\frac{1}{4\sqrt{2}}-\sqrt{2}}{2} = -\frac{9}{8\sqrt{2}}$.

4. naloga (35 točk)

Z enačbama $x^2+y^2+z=5$ in $x^2+y^2-z=3$ je podana krivulja K.

a) Parametriziraj krivuljo K.

b) Določi spremljajoči trieder krivulje K v točki $(2,0,1)$.

c) Izračunaj krivuljni integral vektorskega polja $\vec{F}(x,y,z)=(x^2+y^2+z^2, xz, z)$ po krivulji K, orientirani v pozitivni smeri, na dva načina:

- po definiciji,
- preko Stokesovega izreka.

a) $2z=2 \Rightarrow z=1$

$2(x^2+y^2)=8 \Rightarrow x^2+y^2=4 \Rightarrow x=2\cos\varphi,\ y=2\sin\varphi$

$\vec{F}(\varphi)=(2\cos\varphi, 2\sin\varphi, 1),\quad \varphi\in[0,2\pi)$

b) $(2,0,1)=\vec{F}(0)$

$\vec{F}'(\varphi)=(-2\sin\varphi, 2\cos\varphi, 0) \Rightarrow \vec{F}'(0)=(0,2,0)$

$\vec{T}(0)=\frac{\vec{F}'(0)}{\|\vec{F}'(0)\|}=\frac{1}{2}(0,2,0)=(0,1,0)$

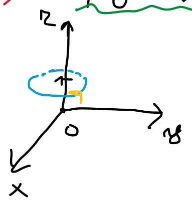
$\vec{F}''(\varphi)=(-2\cos\varphi,-2\sin\varphi,0) \Rightarrow \vec{F}''(0)=(-2,0,0)$

$\vec{F}'(0)\times\vec{F}''(0)=(0,2,0)\times(-2,0,0)=(0,0,4)$

$\vec{B}(0)=\frac{1}{4}(0,0,4)=(0,0,1)$

$\vec{N}(0)=\vec{B}(0)\times\vec{T}(0)=(0,0,1)\times(0,1,0)=(-1,0,0)$

c) PO DEFINICIJI



$\vec{F}(\varphi)=(2\cos\varphi, 2\sin\varphi, 1)$

Ko φ raste od 0 do 2π, se premikamo v pozitivno smer pok.

$\Rightarrow \int_K \vec{F}\cdot d\vec{s}=\int_0^{2\pi}(5,2\cos\varphi,1)\cdot(-2\sin\varphi,2\cos\varphi,0)\,d\varphi=$

$=\int_0^{2\pi}(-10\sin\varphi+4\cos^2\varphi)\,d\varphi=\int_0^{2\pi}(-10\sin\varphi+2+2\cos(2\varphi))\,d\varphi=$

$=(10\cos\varphi+2\varphi+\sin(2\varphi))\Big|_0^{2\pi}=4\pi$

STOKES

Ploskev, katere rob je K, je npr. krog P s parametrizacijo $\vec{F}(r,\varphi) = (r\cos\varphi, r\sin\varphi, 1)$, $r\in[0,2]$, $\varphi\in[0,2\pi)$.

$\vec{F}_r = (\cos\varphi, \sin\varphi, 0)$ $\qquad$ $\vec{F}_\varphi = (-r\sin\varphi, r\cos\varphi, 0)$

$\vec{F}_r \times \vec{F}_\varphi = (0,0,r)$ kaže gor ✓

$\operatorname{rot}\vec{F} = (-x, 2z, z-2y)$

$$\int_K \vec{F}\,d\vec{s} = \int_P \operatorname{rot}\vec{F}\cdot d\vec{S} = \int_0^{2\pi} d\varphi \int_0^2 (-r\cos\varphi, 2, 1-2r\sin\varphi)\cdot(0,0,r)\,dr =$$

$$= \int_0^{2\pi} d\varphi \int_0^2 (r - 2r^2\sin\varphi)\,dr = \int_0^{2\pi} \left(\frac{r^2}{2} - \frac{2r^3}{3}\sin\varphi\right)\Big|_{r=0}^{2} d\varphi =$$

$$= \int_0^{2\pi} \left(2 - \frac{16}{3}\sin\varphi\right) d\varphi = \left(2\varphi + \frac{16}{3}\cos\varphi\right)\Big|_0^{2\pi} = 4\pi$$