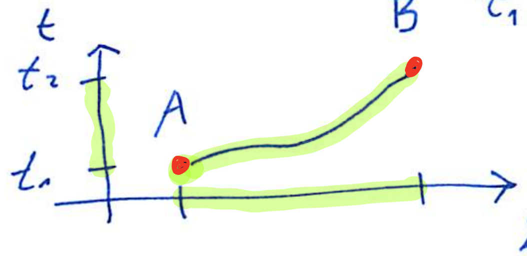
Hamiltonov princip minimalne akcije

• Akcija S

Vpeljemo funkcional (ki preslika funkcijo v $\mathbb{R}$),

$$\underset{\text{akcija}}{S} = S[\{q_i(t)\}, \{\dot{q}_i\}] = \int_{t_1}^{t_2} \underbrace{L(q_i(t), \dot{q}_i(t), t)}_{\text{funkcija za } i=1,\dots,n}\, dt \in \mathbb{R}$$



$q_i(t)$

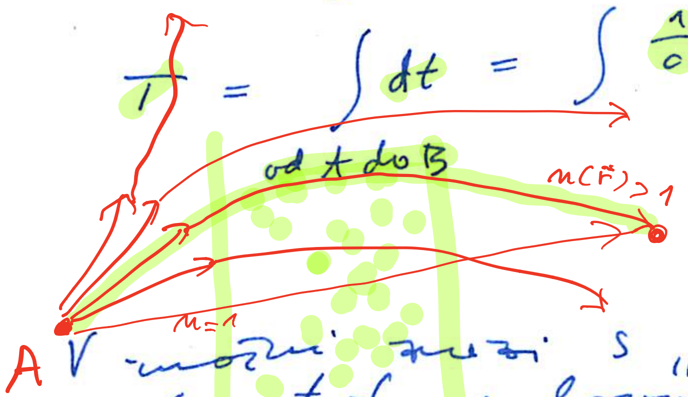
Iščemo tako poti $q_i(t)$, da bo S minimalen. Posledica so tiri, ki so rešitev E-L enačb. To je „Hamiltonov princip minimalne akcije". Kot bomo že dokazali.

Podoben je „Fermatov" princip, ki pravi, da se (EM) valovanje širi tako, da „žarek" pride na cilj v minimalnem optičnem času oz. da je „faza" minimalna v nehomogenem sredstvu,

$$T = \int dt = \int \frac{1}{c}\frac{c}{v}\frac{ds}{dt}\, dt = \frac{1}{c}\int_A^B n(s)\, ds$$

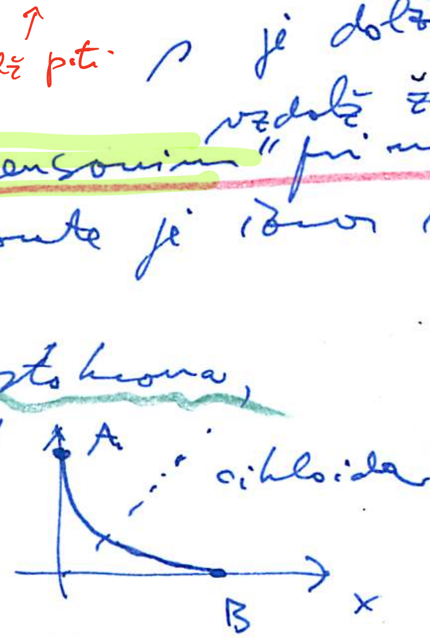
$\frac{c}{v} = n(s)$, $\frac{ds}{dt}dt = ds$ vzdolž poti, φ = ekstrem

s je dolžina poti vzdolž žarka.



V množici zvezi s „Huygensovim" principom: vsaka točka valovne fronte je izvor valovanja s hitrostjo c. Optika.

Zgodovinsko: brahistohrona,

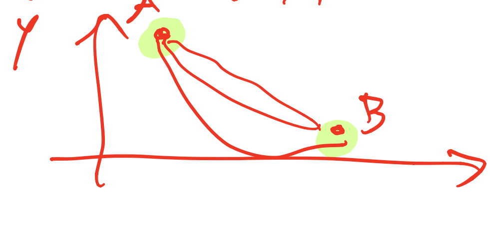


Variacijsko iskanje minimuma

$y(x) = ?$

$$I = \int_{x_1}^{x_2} f(y, y', x)\, dx = \text{ekstremum}, \qquad y' = \frac{dy}{dx}$$

odvisno od problema: $y(x)$, $y'(x)$ npr. brahistohrona



na robu je

$y(x_1) = y_1$

$y(x_2) = y_2$.

Nastavek,

$y'(x,\alpha) = y'(x,0) + \alpha\eta'(x)$; $y(x,0) = y(x)$,

$\eta(x)$ je poljubna gladka funkcija, na robu $\eta(x_1) = \eta(x_2) = 0$. Torej je $I = I(\alpha)$ in iščemo minimum,

$$\frac{dI(\alpha)}{d\alpha} = \int_{x_1}^{x_2} \Big(\frac{\partial f}{\partial y} \underbrace{\frac{\partial y}{\partial \alpha}}_{\eta(x)} + \frac{\partial f}{\partial y'} \frac{\partial y'}{\partial \alpha} \Big) dx = 0$$

$$\int_{x_1}^{x_2} \frac{\partial f}{\partial y'} \underbrace{\frac{\partial y'}{\partial \alpha}}_{\eta' = \frac{d\eta}{dx}} dx = \underbrace{\frac{\partial f}{\partial y'} \eta(x) \Big|_{x_1}^{x_2}}_{=0 \text{ po predpostavki}} - \int_{x_1}^{x_2} \eta(x) \frac{d}{dx} \frac{\partial f}{\partial y'} dx$$

Torej,

$$\frac{dI}{d\alpha} = \int_{x_1}^{x_2} \Big(\frac{\partial f}{\partial y} - \frac{d}{dx} \frac{\partial f}{\partial y'} \Big) \eta(x)\, dx = 0,$$

ko velja za $\forall\, \eta(x)$, zato

$$\boxed{\frac{d}{dt} \frac{\partial L}{\partial \dot y} - \frac{\partial L}{\partial y} = 0.}$$

$x \to t$

Variacijska izpeljava Euler-Lagrangevih enačb

Sedaj rabe to enostavno ponovimo za q_i in S, $\qquad \dot{q}_i \qquad \delta q_i = \alpha \eta_i(t)$

$$q_i(t,\alpha) = q_i(t) + \alpha \eta_i(t) = q_i(t) + \delta q_i(t)$$

$q_i(t)$ — iskana rešitev; $\delta q_i(t)$ — variacija

Točno tako kot prej je $\delta q_i(t_1) = \delta q_i(t_2) = 0$,

$$S \to \quad \delta S = \int_{t_1}^{t_2} \Big(\sum_i \frac{\partial L}{\partial q_i} \delta q_i + \underbrace{\sum_i \frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i}_{*} \Big) dt = 0 \qquad (\alpha \to 0)$$

$$* \int_{t_1}^{t_2} \frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i \, dt = \underbrace{\frac{\partial L}{\partial \dot{q}_i} \delta q_i \Big|_{t_1}^{t_2}}_{=0} - \int_{t_1}^{t_2} \delta q_i \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} \, dt \, ,$$

$$\delta S = \int_{t_1}^{t_2} \sum_i \Big(\frac{\partial L}{\partial q_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} \Big) \delta q_i \, dt = 0 \quad \text{za } \forall \, \delta q_i$$

$$\Rightarrow \boxed{\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0 .}$$

Hamiltonov princip

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spomnimo se $\quad L' = L + \frac{d}{dt} F(q_i, t) \qquad q_i + \delta q_i$

$$\delta \quad S' = \int_{t_1}^{t_2} L' dt = \int_{t_1}^{t_2} L \, dt + \int_{t_1}^{t_2} \frac{d}{dt} F \, dt = S + F(q_i(t), t) \Big|_{t_1}^{t_2}$$

$$\delta S' = \delta S + \underbrace{\delta F \Big|_{t_1}^{t_2}} \qquad \Rightarrow \boxed{\delta S' = \delta S} \quad \text{pri istih } q_i(t).$$

$$\delta F \Big|_{t_1}^{t_2} = \sum_i \frac{\partial F}{\partial q_i} \underbrace{\delta q_i(t) \Big|_{t_1}^{t_2}}_{=0} = 0$$

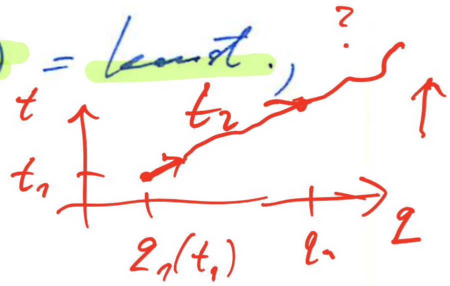
Železni repertoar: centralni potencial, vrtavka, majhna nihanja.

Enodimenzionalni primeri,

$$L = \frac{1}{2} m(q)\,\dot q^2 - U(q).$$

Vemo že, da velja

$$E = T + U = \frac{1}{2} m(q)\,\dot q^2 + U(q) = \text{konst.},$$



torej

$$\frac{dq}{dt} = \dot q = \sqrt{\frac{2(E-U(q))}{m(q)}}$$

$$t_2 - t_1 = f(q_2) \quad ??$$

in

$$\int_{t_1}^{t_2} dt = \int_{q_1}^{q_2} \sqrt{\frac{m(q)}{2(E-U(q))}}\, dq \Rightarrow \boxed{q_2(t_2)} \text{ pri}$$

začetnem pogoji $q(t_1) = q_1$

Torej je tu skrčimo problem vedno rešljiva z direktno integracijo. Jasno, da ni vedno elementarno rešljivo, numerično pa.

Primer: $q = x$, $m(q) = m$ masa

$\dot x(t_1)$? $\pm$

$x(t_1)$

$$E = \frac{1}{2} m v^2 + U(x), \qquad v^2 \geq 0$$

?

$$\pm \frac{dx}{dt} = \sqrt{2(E-U(x))/m}$$

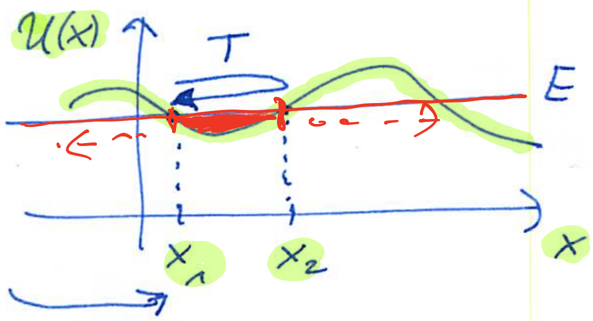
dve rešitvi

$$\boxed{\pm t = \sqrt{\frac{m}{2}}\left|\int_{x_1}^{x(t)} \frac{dx}{\sqrt{\underbrace{E-U(x)}_{\geq 0}}}\right|} \quad \text{pri } x(0) = x_1.$$

nihajni čas: T_0

$$T_0 = 2\sqrt{\frac{m}{2}}\left|\int_{x_1(E)}^{x_2(E)} \frac{dx}{\sqrt{E-U(x)}}\right|$$

obračalni točki



konkretno: navadno nihalo

$V = mgh$, $h \propto \cos\vartheta + \ldots$

$$L = \frac{1}{2} m l^2 \dot{\varphi}^2 + mgl\cos\varphi \qquad = \text{konst.}$$

$$E = \frac{1}{2} m l^2 \dot{\varphi}^2 - mgl\cos\varphi = E_0 = -mgl\cos\varphi_0$$

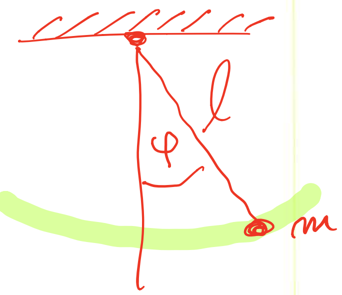
$(\varphi = \varphi_0$ in $\dot{\varphi} = 0)$.

Nihajni čas T_0 je 4 x čas od $\varphi = 0$ do $\varphi = \varphi_0$,

$$\sin^2\frac{\varphi}{2} = \frac{1}{2}(1-\cos\varphi)$$

$$\rightarrow \quad T_0 = 4\sqrt{\frac{l}{2g}}\left|\int_0^{\varphi_0}\frac{d\varphi}{\sqrt{\cos\varphi - \cos\varphi_0}}\right| =$$

$$= 2\sqrt{\frac{l}{g}}\int_0^{\varphi_0}\frac{d\varphi}{\sqrt{\sin^2\frac{\varphi_0}{2} - \sin^2\frac{\varphi}{2}}} =$$



$$= 4\sqrt{\frac{l}{g}}\, K\big(\underbrace{\sin\tfrac{1}{2}\varphi_0}_{k}\big),$$ kjer je

$$\boxed{K(k) = \int_0^{\frac{\pi}{2}}\frac{d\xi}{\sqrt{1-k^2\sin^2\xi}}} \quad ; \sin\xi = \frac{\sin\varphi/2}{\sin\varphi_0/2}$$

popolni eliptični integral I. vrste.

Če $k \ll 1$, $\sin\frac{\varphi_0}{2} \approx \frac{\varphi_0}{2}$, velja

$$\boxed{T = 2\pi\sqrt{l/g}\left(1 + \frac{\varphi_0^2}{16} + \mathcal{O}(\varphi_0^4)\right)}.$$

in tudi za $\varphi, \varphi_0 \ll 1$,

$$T_0 = 2\sqrt{\frac{l}{g}}\int_0^{\varphi_0}\frac{d\varphi}{\sqrt{\frac{\varphi_0^2}{4} - \frac{\varphi^2}{4}}} = 4\underbrace{\sqrt{\frac{l}{g}}}_{\frac{1}{\omega}}\int_0^{\varphi_0}\frac{d\varphi}{\sqrt{\varphi_0^2-\varphi^2}} = \frac{4\cdot 1}{\omega}\arcsin\varphi/\varphi_0 = \frac{4\cdot 1}{\omega}\arcsin 1 = \frac{4\cdot 1}{\omega}\frac{\pi}{2}.$$

$$\Rightarrow \boxed{T_0 = 4\cdot\sqrt{\frac{l}{g}}\,\frac{\pi}{2} = 2\pi\sqrt{\frac{l}{g}} = \frac{2\pi}{\omega}}$$

$$\int\frac{dx}{\sqrt{a^2-x^2}} = \arcsin x/a$$

Osnovni primer: harmonski oscilator

$\ddot{x} + \omega^2 x = 0$ $\quad x(t) = ? \quad = A \cos\omega t + B \sin\omega t$ ✓

$$\boxed{E = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} k x^2 = E_0 = \text{konst.}}$$

E_0 je konstanta, odvisna od začetnih pogojev.

najprej izrazimo hitrost $\dot{x}(t)$,

$$\dot{x}(t) = \frac{dx}{dt} = \pm \sqrt{\frac{2E_0 - x(t)^2}{m}}$$

predznak glede na začetne pogoje (smer hitrosti)

Primer: $x(0) = x_0$

$\dot{x}(0) = 0 \quad ; \quad \omega^2 = \frac{k}{m}$

$$t\omega = \omega \int_0^t dt = - \int_{x_0}^{x(t)} \frac{dx}{\sqrt{x_0^2 - x^2}} = -\arcsin \frac{x(t)}{x_0} - \underbrace{\arcsin 1}_{\pi/2}$$

$\begin{cases} x(0) = x_0 \\ \dot{x}(0) = 0 \end{cases}$

$$x_0 \sin\left(\frac{\pi}{2} - \omega t\right) = x(t)$$

$\hookrightarrow \sin\frac{\pi}{2} \cos\omega t - \cos\frac{\pi}{2} \sin\omega t$

$$\Rightarrow \quad \boxed{x(t) = x_0 \cos\omega t.}$$

Analogno se postopa pri drugih začetnih pogojih.

$\sin(\alpha \pm \beta) = \sin\alpha \cos\beta \pm \cos\alpha \sin\beta$