"Problem" dveh teles

se rodno zreducira na eno telo, kot sledi:

$$T = \frac{1}{2} m_1 |\dot{\vec{r}}_1|^2 + \frac{1}{2} m_2 |\dot{\vec{r}}_2|^2, \quad V = U(\vec{r}, \dot{\vec{r}}),$$

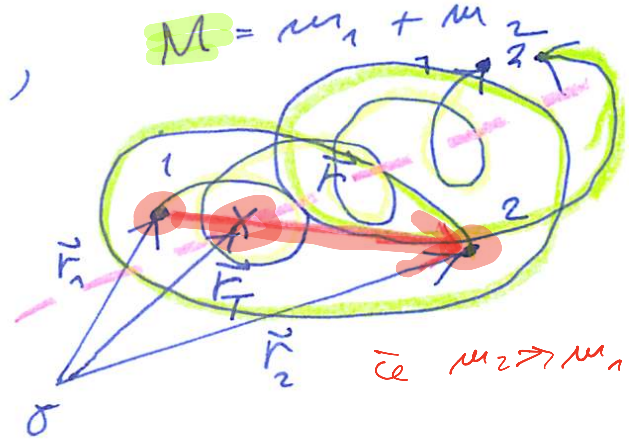
$$\vec{r} = \vec{r}_2 - \vec{r}_1$$ relativna koordinata

6 leg + 6 hitrosti

D.N.

Težišče,

$$M \vec{r}_T = m_1 \vec{r}_1 + m_2 \vec{r}_2, \qquad M = m_1 + m_2$$



$$\vec{r}_1 = \vec{r}_T - \frac{m_2}{M} \vec{r}$$

$$\vec{r}_2 = \vec{r}_T + \frac{m_1}{M} \vec{r}$$

če $m_2 \gg m_1$

$$T = \frac{1}{2} M |\dot{\vec{r}}_T|^2 + \frac{1}{2} m |\dot{\vec{r}}|^2 ; \quad m = \frac{m_1 m_2}{M} \approx m_1$$

$$\frac{1}{m} = \frac{1}{m_1} + \frac{1}{m_2}$$

reducirana masa

če m_1 ali $m_2 \to \infty$, je $m = m_2$ ali m_1.

$$\boxed{L = \frac{1}{2} M |\dot{\vec{r}}_T|^2 + \frac{1}{2} m |\dot{\vec{r}}|^2 - U(\vec{r}, \dot{\vec{r}})}$$

- konstante

$\vec{r}_T$ ciklična $\Rightarrow \frac{d}{dt} \frac{\partial L}{\partial \dot{\vec{r}}_T} = 0 \Rightarrow \boxed{\vec{p}_T = M \dot{\vec{r}}_T = \text{konst.}}$

$$\vec{r}_T(t) = \vec{r}_T(0) + \dot{\vec{r}}_T(0)\, t$$

začetni pogoji: 3 + 3.

ostanejo 3+3:

$$L = \frac{1}{2} m |\dot{\vec{r}}|^2 - U(\vec{r}, \dot{\vec{r}}).$$

$$E = \frac{1}{2} m |\dot{\vec{r}}|^2 + U = \text{konst.}$$

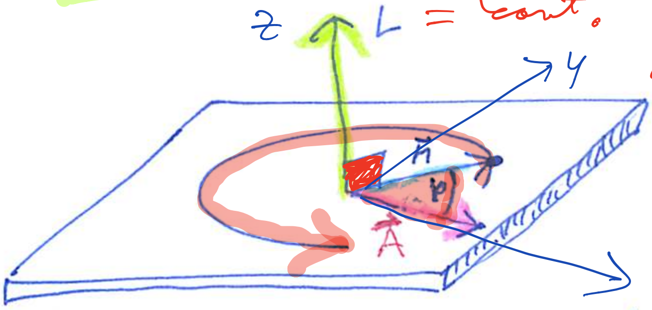
če je sila centralna, $V(\vec r) = V(|\vec r|) = V(r)$,

$$\vec F = -\nabla V = -\frac{dV(r)}{dr}\frac{\vec r}{r},$$

$$\dot{\vec L} = \vec M = \vec r \times \vec F = 0, \quad \vec L = \text{konst.} \quad (\text{že vemo})$$

Torej lahko definiramo ravnino,

$\vec L \cdot \vec r = 0$, stalno ravnino (x, y),



$$\vec L = L\,\hat e_z$$

Vpeljemo še znane polarne koordinate v tej ravnini; r, φ, $T = \frac{1}{2}\sum_{jk} m_{jk}(\underline{q})\dot q_j \dot q_k$

1D → $L(r)$ ✓

$$L = \frac{1}{2}m(\dot r^2 + r^2\dot\varphi^2) - V(r), \quad \varphi \text{ ciklična},$$

$$p_\varphi = \frac{\partial L}{\partial \dot\varphi} = m r^2 \dot\varphi = l = l_0 = \text{konst.}$$

V času Δt opiše $\vec r$ lok oz. ploščino P,

$$\Delta P = \frac{1}{2} r \cdot r\Delta\varphi = \frac{1}{2} r^2 \dot\varphi \Delta t \Rightarrow \frac{dP}{dt} = \boxed{\frac{1}{2} r^2 \dot\varphi = \text{konst.}} \propto l$$

2. Keplerjev zakon

(velja za $\forall\, V(r)$.)

vaje: $V(r) \longrightarrow$ orbite

Ali so še kakšne konstante gibanja?

Za primer $V \propto \frac{1}{r}$ se ohranja

Laplace-Runge-Lenzov vektor Pauli

neodvisen od teh ga ni odkril (gl. wiki)

simetrija: delec se giblje po 4D hipersferi; QM.

Ponovimo... Torej,

$\vec{r}_T$ je očitno ciklična koordinata in zato,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}_T} = 0 \Rightarrow M\dot{\vec{r}}_T = M\vec{v}_T = \vec{P}_T = \text{konst.}$$

gibalna količina.

Torej zadošča, če vzamemo le enodelčni problem z

$$L = \frac{1}{2} m |\dot{\vec{r}}|^2 - U(\vec{r}, \dot{\vec{r}}).$$

Konstante gibanja (razen $\vec{P}_T$ in celotna L in E).

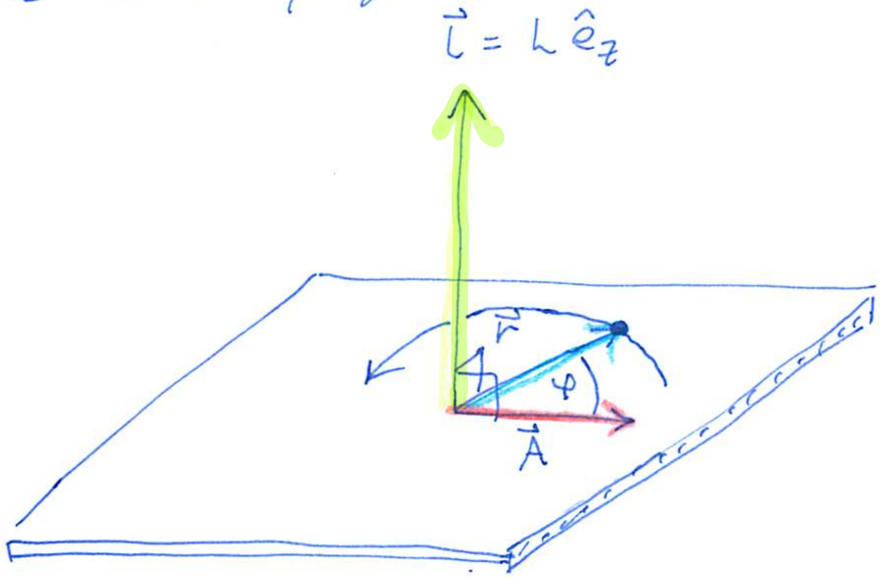
Obravnavamo torej točkast delec v izotropnem danem potencialu $V(|\vec{r}|)$,

$$\vec{F} = -\nabla V(r) = -\frac{\partial V}{\partial r}\frac{\vec{r}}{r},$$

$$\vec{M} = \vec{r} \times \vec{F} = 0 \Rightarrow \dot{\vec{L}} = \vec{M} = 0,$$

- $\vec{L} = \vec{r} \times \vec{p} = \text{konst.}$; $\vec{r}\cdot\vec{L} = \vec{r}\cdot(\vec{r}\times\vec{p}) = 0$; $\vec{p} = m\dot{\vec{r}}$

Delec se torej giblje v ravnini $\perp \vec{L}$.



- Seveda se ohranja tudi celotna energija

Laplace-Runge-Lenzov vektor

naj velja

$$m\ddot{\vec r} = \vec F(\vec r) = f(r)\frac{\vec r}{r}\,; \qquad f(r) = -\frac{\partial V(r)}{\partial r}$$

$$\dot{\vec p} = m\ddot{\vec r} \qquad \times \vec L$$

$$\dot{\vec p}\times\vec L = \frac{m f(r)}{r}\left(\vec r\times(\vec r\times\dot{\vec r})\right) =$$

$$= \frac{m f(r)}{r}\left[\vec r\,(\vec r\cdot\dot{\vec r}) - \dot{\vec r}\,r^2\right]\left(\frac{r^3}{r^3}\right)$$

kot že vemo, $\frac{d}{dt}r^2 = 2r\dot r = \frac{d}{dt}\vec r\cdot\vec r = 2\,\vec r\cdot\dot{\vec r}$

torej

$$\frac{d}{dt}(\vec p\times\vec L) = \dot{\vec p}\times\vec L + \vec p\times\underbrace{\dot{\vec L}}_{=0} =$$

$$= -\frac{m f(r)}{r}\left(\frac{\dot{\vec r}}{r} - \frac{\dot r\,\vec r}{r^2}\right)r^3 =$$

$$= -m f(r)\,r^2\,\frac{d}{dt}\left(\frac{\vec r}{r}\right)$$

Če velja (npr. gravitacija): $-f(r)r^2 = k\;(= mMG)$ konstanta, $\quad f \propto \frac{1}{r^2}$

$$\Rightarrow \frac{d}{dt}(\vec p\times\vec L) = \frac{d}{dt}\,mk\frac{\vec r}{r}$$ in zato očitno

$$\boxed{\vec A = \vec p\times\vec L - mk\frac{\vec r}{r} = \text{konst.}} \quad \text{oz.} \quad \dot{\vec A} = 0.$$

$k = mMG$ ← gravitacijska konstanta

m ~ masa Zemlje, M ~ masa Sonca (v limiti $m/M \to 0$).

$\vec A = \vec A_0$ je torej konstanta gibanja pri Keplerjevem problemu, $V \propto \frac{1}{r}$.

$\vec a\times(\vec b\times\vec c) = (\vec a\cdot\vec c)\,\vec b - (\vec a\cdot\vec b)\,\vec c$

P.N.?

$=\dots$

$\frac{e_1 e_2}{4\pi\varepsilon_0}$?

Očitno velja

$$\vec A\cdot\vec L = \underbrace{(\vec p\times\vec L)\cdot\vec L}_{=0} - mk\,\frac{\overbrace{\vec r\cdot\vec L}^{=0}}{r} = 0. \qquad \vec A \perp \vec L$$

Torej je $\vec A$ v ravnini $\perp$ na $\vec L$, kot $\vec r$. Konkretna vrednost $\vec A$ je odvisna od začetnih pogojev. Imamo torej

$$3(\vec L) + 3(\vec A) + 1(E) + 3(\vec P_T) = 7+3$$

$= 10$ konstant gibanja (niso neodvisne).

Konstante gibanja bistveno poenostavijo reševanje (brez zapor — omogočijo). Tir določimo tako:

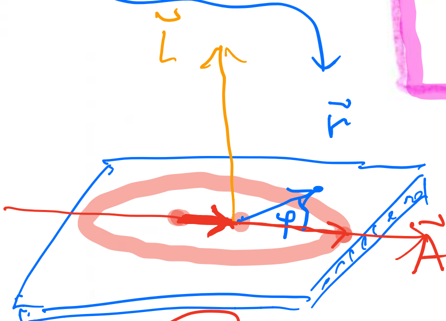
$$\vec A\cdot\vec r = A r\cos\varphi = \underbrace{\vec r\cdot(\vec p\times\vec L)}_{\vec L\cdot(\vec r\times\vec p) = L^2 \equiv l^2 = \text{konst.}} - mkr$$

$\Rightarrow r(\varphi)$

$$r(A\cos\varphi + mk) = l^2$$

$$r(\varphi) = \boxed{r = \frac{l^2}{mk\left(1+\frac{A}{mk}\cos\varphi\right)} = \frac{r_0}{1+\varepsilon\cos\varphi}}$$

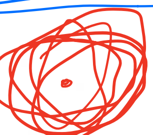
$\pm$ (glede na def. kota)

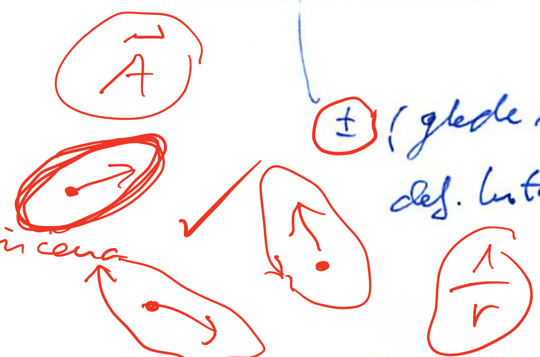


$\vec A$ = konst.

$$\varepsilon = \frac{A}{mk}$$

$$r_0 = \frac{l^2}{mk}$$





zaključena

splošno: ni zaključena orbita če ni $V \propto \frac{1}{r}$

$\Rightarrow$ stožnica; $A \neq 0$

Če $A=0$, je to posebni pad in je orbita krožnica (trivialno vidno).

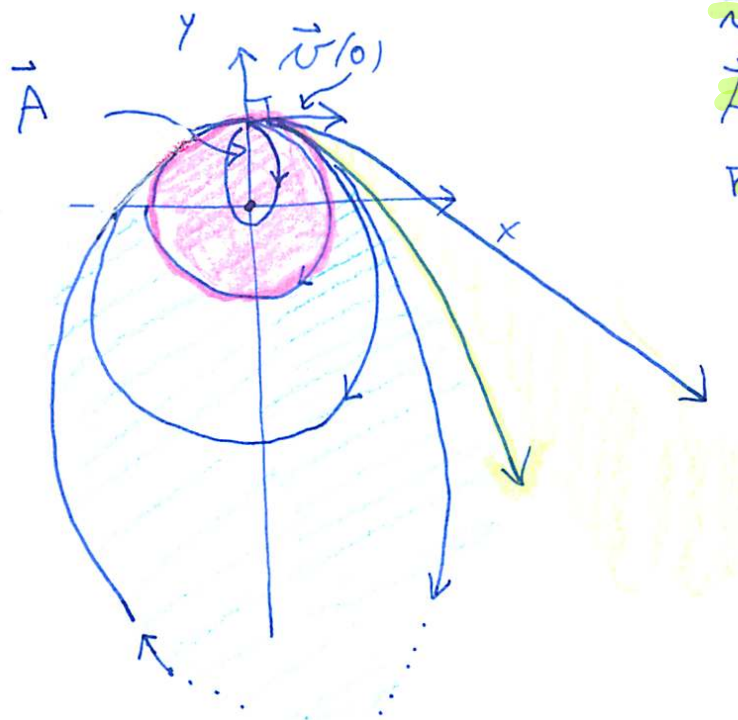
Orbite pri Keplerjevem problemu

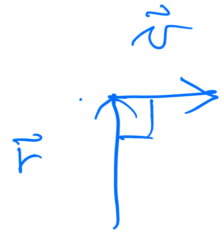
Izberemo si začetne pogoje takole:

$$\vec{v}(0) = (v, 0, 0),\ v \neq 0$$

$$\vec{A} = (0, A, 0)$$

$$\vec{r}(0) = (0, R, 0)$$

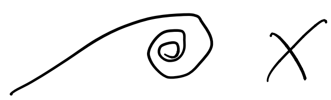




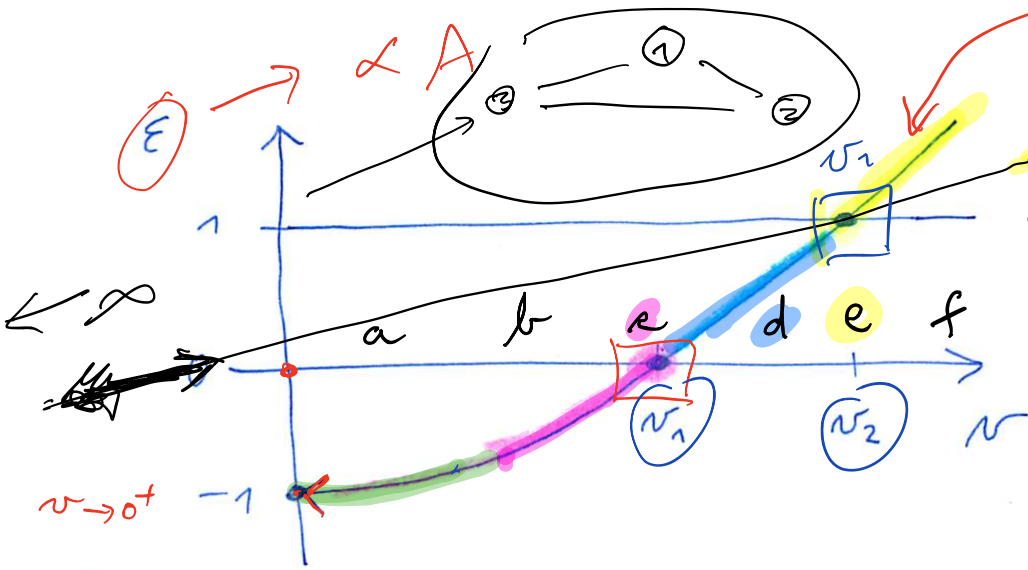
$$\vec{A} = m\vec{v} \times (\vec{r} \times m\vec{v}) - mk\frac{\vec{r}}{r}$$

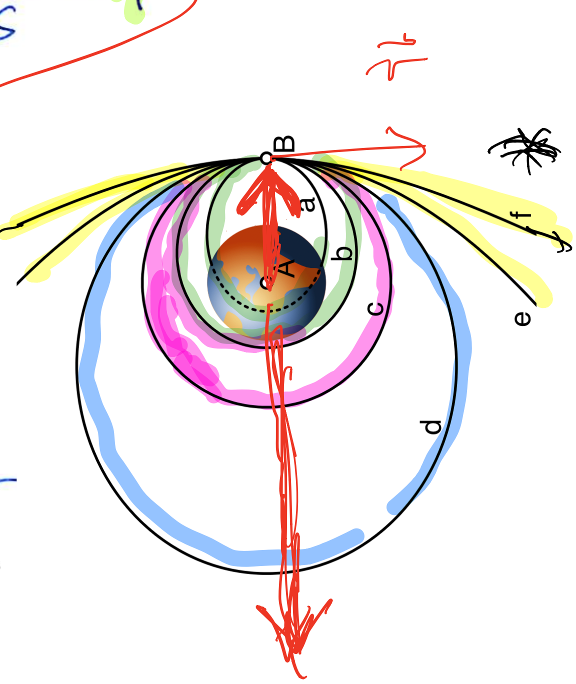
$\frac{1}{r}$

$$\pm|\vec{A}| = m^2 R v^2 - m^2 MG$$



$$\varepsilon = \frac{A}{mk} = \frac{mRv^2}{mMG} - 1 = \frac{Rv^2}{MG} - 1$$





(a)

$\varepsilon = 0$: $\quad v_1 = \sqrt{\frac{MG}{R}}$ krožnica

Energija,

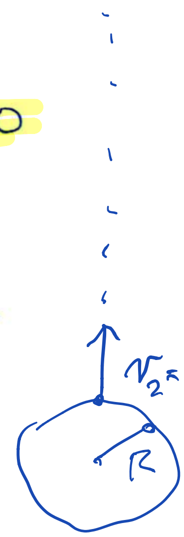
$$E_1 = \frac{1}{2} m v_1^2 - \frac{mMG}{R} = \boxed{-\frac{1}{2}\frac{mMG}{R}} < 0$$

virialni izrek:

$$\langle T \rangle = \frac{k}{2} \langle V \rangle$$

$k = -1$

O.N.
$\langle E \rangle = \langle T \rangle + \langle V \rangle$
$2\langle T \rangle = k \langle V \rangle$
$\Longrightarrow \langle T \rangle = f(E)$
$\langle V \rangle = g(E)$



(b)

$\varepsilon = 1$: $\quad v_2 = \sqrt{2 \frac{MG}{R}}$

$E_2 = 0$

hiperbola za večje hitrosti; če $v = v_2$ je parabola („elipsa z enim goriščem $\to \infty$").

(c) $\varepsilon \geq 1$ in $v_3 \geq v_2$,

$E > 0$ virialni teorem ne velja
neomejeno gibanje

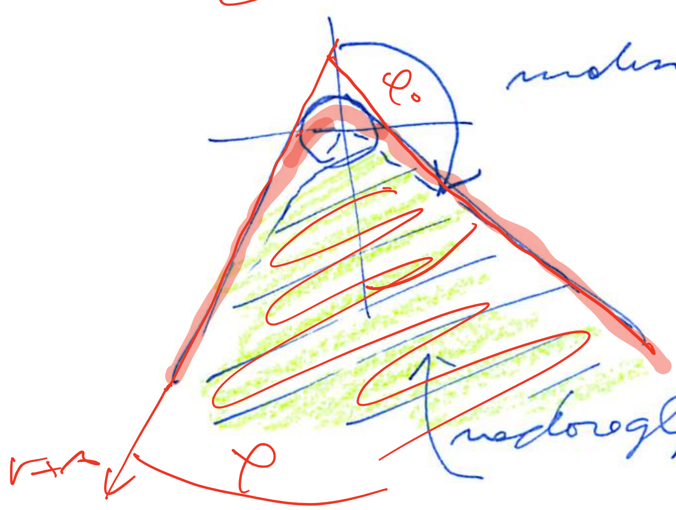
$$r(\varphi) = \frac{r_0}{1 + \varepsilon \cos \varphi}$$

$1 - |\varepsilon \cos \varphi_0|$

$r \to \infty \Longleftarrow \boxed{1 \pm \varepsilon \cos \varphi_0} = 0 \Rightarrow \cos \varphi_0 = \frac{1}{\varepsilon}$ (oz. $\varphi_0 + 2\pi$)

maksimalni kot φ_0

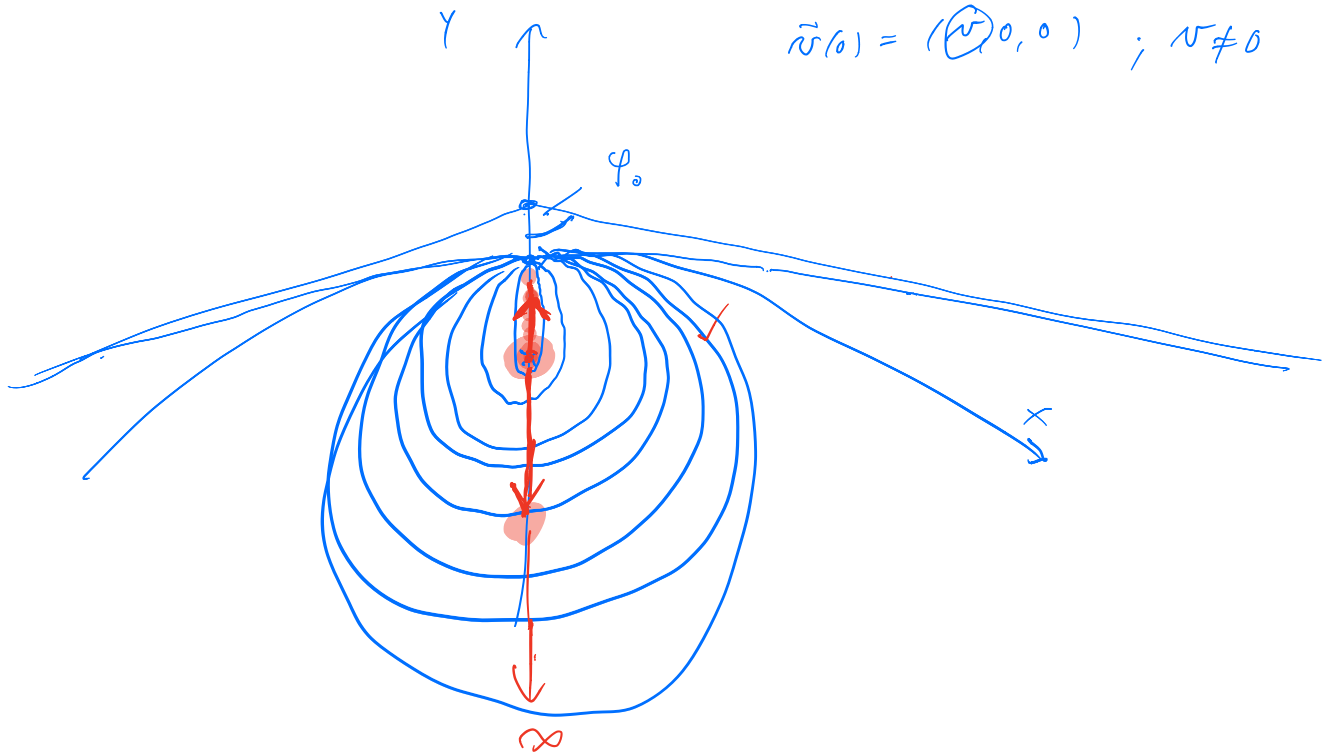
π



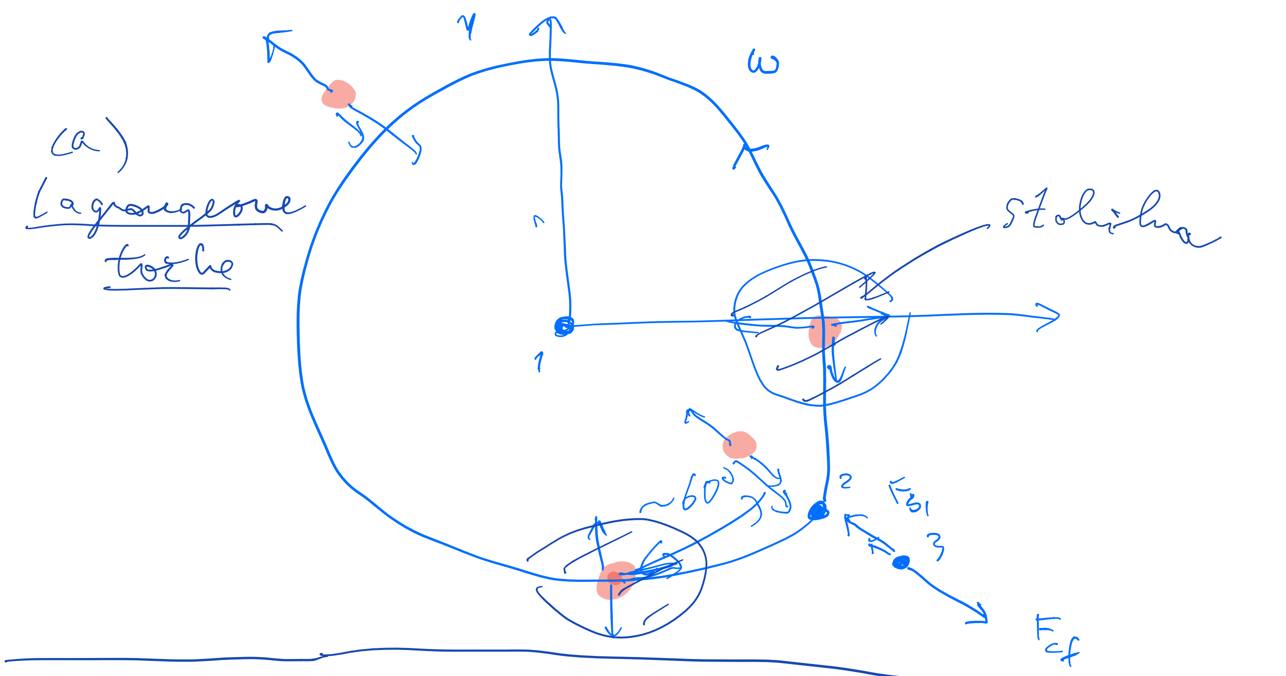
nedosegljivo ($\cos \varphi > 1$)

opomba: pri $\varepsilon = 1$ se smer vektorja $\vec{A}$ ohrani (s spremembo predznaka); vloga gorišč se zamenja.

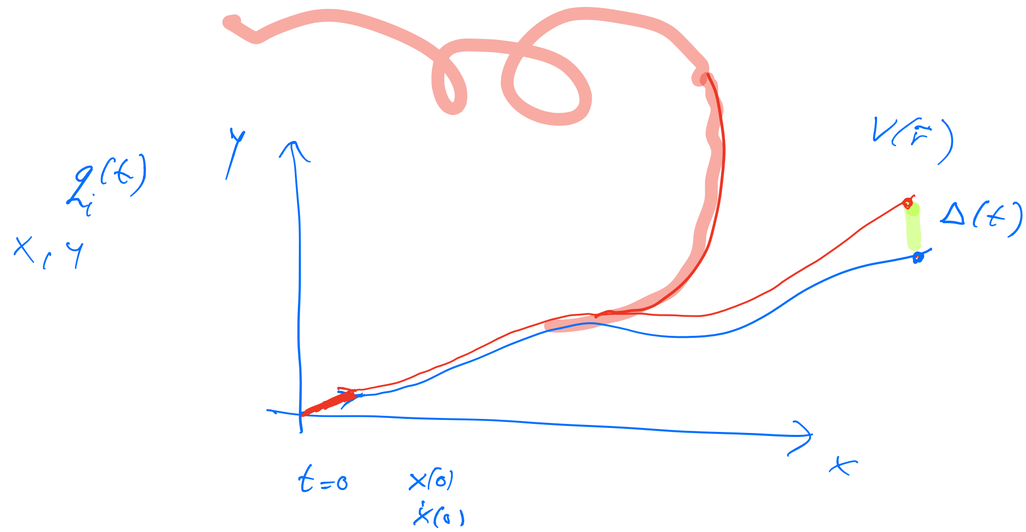
$\vec{v}(0) = (v, 0, 0)$; $v \neq 0$



(a) Lagrangeove točke



(b) Kaos

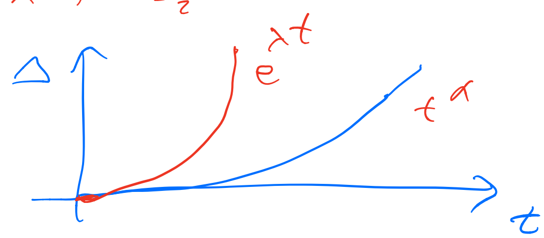


$t=0 \quad x(0) + \varepsilon_1 \qquad \vec{F} \to \vec{\varepsilon}$

$\dot{x}(0) + \varepsilon_2$

• $\Delta(t) \to \varepsilon t^{\alpha}$

NI KAOS



•• $\Delta t| \to \varepsilon e^{\lambda t}$ KAOS $\lambda \neq 0$

Lyapunovov eksponent