Opis lege togega telesa: Eulerjevi koti

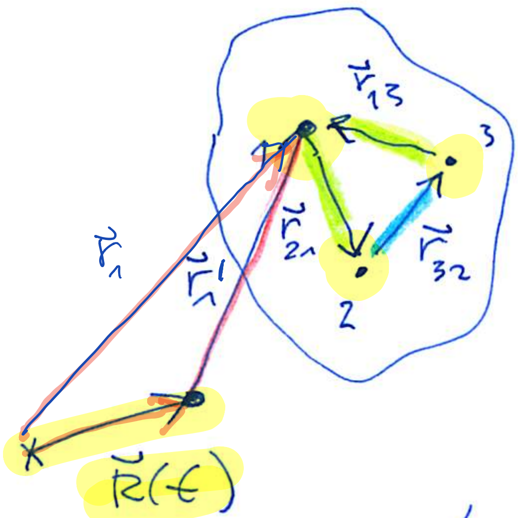
Togo telo ima razdaljo med vsemi dvema točkama konstantno,

$$|\vec r_i - \vec r_j| = c_{ij} = \text{konst.}$$

Lega je podana s 6 neodvisnimi koordinatami,

- $\vec r_1$ (3x)
- $\vec r_{21}$ (2x)
- $\vec r_{32}$ oz. $\vec r_{13}$ (1x)

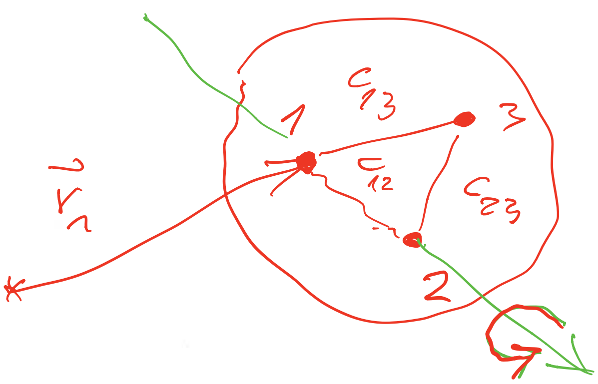
$\rightsquigarrow$ 6x



Običajno vpeljemo gibajoči se koordinatni sistem,

$$\vec r_i = \vec R + \vec r_i'$$

Med $\vec r_i$ in $\vec r_i'$ se transformira z linearno preslikavo (matrika).



$3 + 2 + 1 = 6$

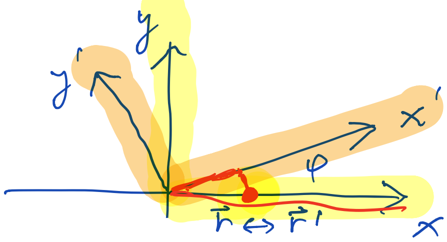
(x, y, z) — 1

(ϑ, φ) — 2

ψ — 3

Rotacija koordinatnega sistema (telesa)

(a) Pasivna rotacija:

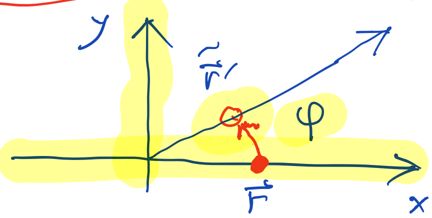


$$\begin{pmatrix}\cos\varphi\\ -\sin\varphi\end{pmatrix} = \begin{pmatrix}\cos\varphi & +\sin\varphi\\ -\sin\varphi & \cos\varphi\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix}$$

$\vec r\,' = R_3(\varphi)\,\vec r$ R_3 – pasivna

ista točka v prostoru, zapisana v novem sistemu

(b) Aktivna rotacija:



$$\begin{pmatrix}\cos\varphi\\ \sin\varphi\end{pmatrix} = \begin{pmatrix}\cos\varphi & -\sin\varphi\\ +\sin\varphi & \cos\varphi\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix}$$

$\tilde{\vec r}\,' = \tilde R_3(\varphi)\,\vec r = R_3(-\varphi)\,\vec r$

zarotiran in zapisan v starem sistemu $\tilde R_3$ – aktivna

(c) Rotacija je ortogonalna transformacija

$$\vec a\,'\cdot\vec b\,' = R\vec a\cdot R\vec b = \vec a\cdot R^T R\,\vec b = \vec a\cdot\vec b$$

$$R^T R = R R^T = I \quad , \quad R^{-1}(\varphi) = R^T(\varphi) = R(-\varphi)$$

$$R = R_2 R_1 \quad , \quad R^{-1} = R_1^{-1} R_2^{-1} = R_1^T R_2^T = R^T$$

produkt ortogonalnih je ortogonalna

(d) nekatere lastnosti determinant

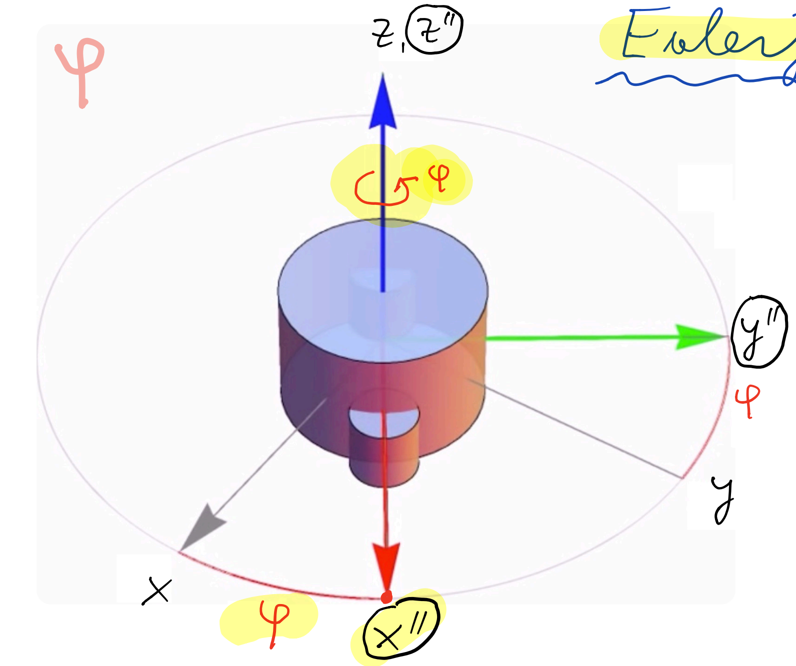
$$\det AB = \det A \det B$$

$$\det R^T = \det R \qquad \text{ker } R^T = R^{-1}$$

$$\det R^T R = \det R^T \det R = \det I = 1 \Rightarrow \det R = \pm 1$$

$$\det R^T = \frac{1}{\det R}$$

Eulerjevi koti $(\varphi, \vartheta, \psi)$



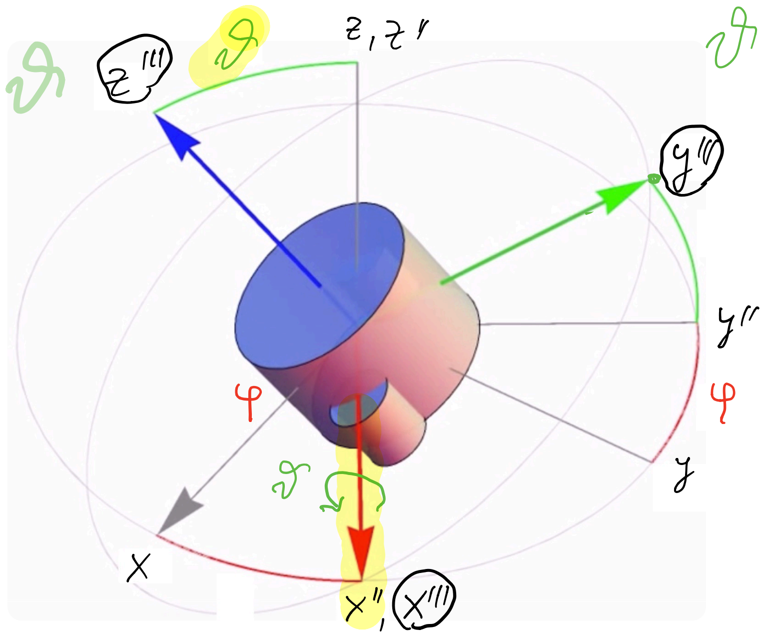
rotacija okoli z za φ

$\vec{r} \to \vec{r}\,''$ precesija

$$T(\varphi) = \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$T(\varphi) = R_3(\varphi) = T^{-1}(-\varphi)$$

$$T^{-1} = T^T$$

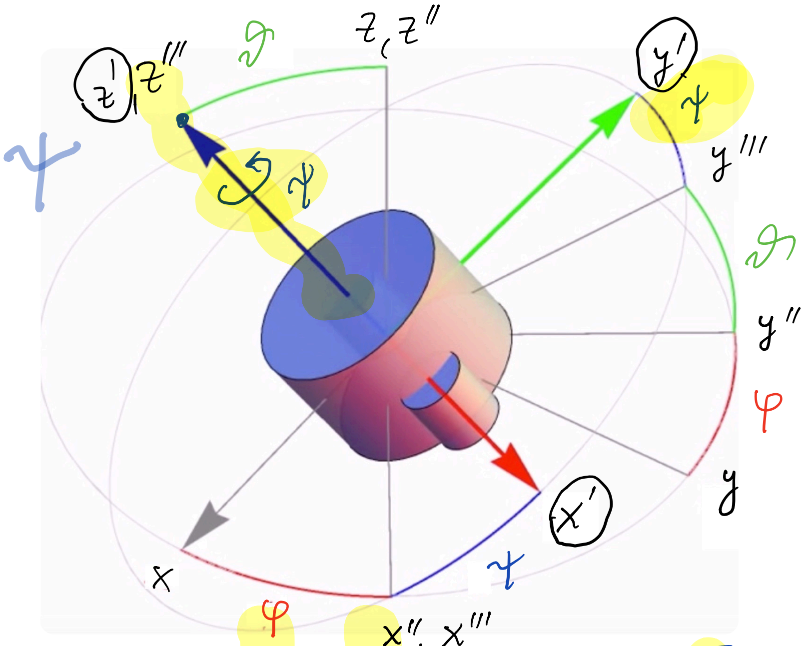


rotacija okoli x'' za ϑ

$\vec{r}\,'' \to \vec{r}\,'''$ nutacija

$$U(\vartheta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\vartheta & \sin\vartheta \\ 0 & -\sin\vartheta & \cos\vartheta \end{pmatrix}$$

$$U(\vartheta) = R_1(\vartheta), \quad U^{-1} = U^T$$



rotacija okoli z'' za ψ

$\vec{r}\,''' \to \vec{r}\,'$ rotacija

$$\vec{r} \longrightarrow \vec{r}\,' = R\,\vec{r}$$

$$V(\psi) = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$V(\psi) = R_3(\psi), \quad V^{-1} = V^T$$

$$R = V(\psi)\,U(\vartheta)\,T(\varphi); \quad R^T = T^T U^T V^T = R^{-1}$$

Eulerjev izrek o rotacijah
(1775)

$R^T = R^{-1}$

ortogonalna

Naj bo R rotacijska matrika, ki transformira $\vec r \to \vec r'$, ?

$$\boxed{R}\,\vec r = \vec r'.$$

Eulerjev izrek:

za $\forall R$ $\exists\, \vec n \neq 0$: $R\vec n = 1\vec n$, torej $\exists$ lastni vektor z lastno vrednostjo 1.

Dokaz:

Rotacijske matrike so ortogonalne, velja

transponirana

$$R^{-1} = R^T \quad \text{oz.} \quad RR^T = R^T R = I.$$

Determinanta je ± 1:

$$1 = \underbrace{\det RR^T}_{\det I} = \det R^T \det R = (\det R)^2 \Rightarrow \det R = \pm 1.$$

Če $= -1$, je to neprava rotacija, sestavljena iz refleksije in rotacije; refleksija: $\vec r \to -\vec r$.

Dokazati torej želimo, da $\exists\, \vec n$,

$$R\vec n = \vec n = I\vec n$$

$$(R - I)\vec n = 0 \Rightarrow \det(R - I) = 0.$$

Uporabimo dve zvezi;

$$\boxed{\det(-A)} = \det\left[\begin{pmatrix} -1 & & \\ & -1 & \\ & & -1 \end{pmatrix} A\right] = \det\begin{pmatrix} -1 & & \\ & -1 & \\ & & -1 \end{pmatrix} \det A = (-1)^3 \det A = \boxed{-\det A}$$

$$\det A^{-1} = (\det A)^{-1}$$

$\det AB = \det A \det B$

$\det A^T = \det A$

torej

$$\det R^{-1} = 1$$

in

$$\det(R-I) = \det(R-I)^T = \det(R^T - I) =$$
$$= \det(R^{-1} - R^{-1}R) =$$
$$= \det(R^{-1}(I-R)) =$$
$$= \det R^{-1} \det(-(R-I)) =$$
$$= -\det(R-I)$$

$$\Rightarrow \boxed{\det(R-I)=0},$$

torej $\det(R-\lambda I)=0 \Rightarrow \lambda = 1$ lastna vrednost R

$$\Rightarrow (R-I)\vec{n} = 0 \Leftrightarrow \boxed{R\vec{n} = \vec{n}}$$

$\exists\, \vec{n}$ in $\exists\, \varphi_n = ?$; $\varphi_n(\varphi, \vartheta, \psi)$; $\vec{n}(\varphi, \vartheta, \psi)$

Dokazali smo, da je vsaka „rotacijska" matrika, t.j., če $R^{-1} = R^T$ & $\det R = 1$, taka, da ima eno samo os $\vec{n}$, ki je invariantna na R, $R\vec{n} = \vec{n}$; $\vec{n}$ = os rotacije.

Dodatno:

$$\det R = 1 = \underbrace{\lambda_1}_{=1} \lambda_2 \lambda_3 \Rightarrow \lambda_2\lambda_3 = 1$$
$$\Rightarrow \lambda_{2,3} = e^{\pm i\varphi}$$
$$\varphi \in \mathbb{R}.$$