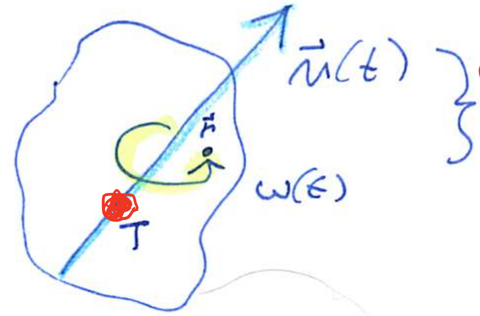
Togo telo z 1 nepremično (fiksno) točko



$\vec{\omega}(t)$ ✓ $\qquad d\vec{\varphi} = \vec{\omega}\, dt$

Točka T naj bo nepremična (translacijo smo odtransformirali). V vsakem trenutku je gibanje telesa po Eulerjevem izreku rotacija okoli neke osi ($\vec{n}$). Za poljubno točko $\vec{r}$ v telesu velja

$$\frac{d\vec{r}}{dt} = \cancel{\left(\frac{d\vec{r}}{dt}\right)_{\text{nein.}} dt} + \vec{\omega}\times\vec{r}\, dt \qquad \vec{\omega}\, dt = d\vec{\varphi}(t)$$

hitrost, kot je v neinercialnem sistemu; $\boxed{\text{pri nas} = 0}$, ker je telo togo in merimo glede na sistem, ki je pripet na telo (trivialno).

torej

$$\vec{v}(t) = \boxed{\vec{\omega}(t)} \times \vec{r}(t) \quad ; \text{ gledamo v inerc. sistemu (laboratorijskem)}$$

($\vec{R}$ = konst.)

• Vrtilna količina telesa (ponovitev)

$$\vec{L} = \sum_{i=1}^{N} \vec{l}_i = \sum_i m_i(\vec{r}_i \times \vec{v}_i) = \sum_i m_i\big(\vec{r}_i \times \underbrace{(\vec{\omega}\times\vec{r}_i)}_{\vec{v}_i}\big)$$

($\vec{\omega}$ v nekem trenutku)

$$\vec{L} = \sum_i m_i\left[(\vec{r}_i\cdot\vec{r}_i)\vec{\omega} - (\vec{r}_i\cdot\vec{\omega})\vec{r}_i\right] = \underline{J}\,\vec{\omega}$$

vztrajnostni moment

Pogosto je ugodno uvesti gostoto mase;

$$\rho(\vec r) = \lim_{\Delta V \to 0} \frac{\Delta m}{\Delta V} \sim \left.\frac{dm}{dV}\right|_{\vec r} ; \quad \Delta m = \sum_{\substack{i \\ \vec r_i \in \Delta V}} m_i .$$

$$M = \sum_i m_i \to \int \rho(\vec r)\, dV .$$

Vztrajnostni moment:

$$\underline{J} = \int \rho(\vec r)\, dV \begin{pmatrix} y^2+z^2 & -xy & -xz \\ -yx & x^2+z^2 & -yz \\ -zx & -zy & x^2+y^2 \end{pmatrix}$$

realen

simetričen

oz.: $$\boxed{J_{\alpha\beta} = \int \rho(\vec r)\left(r^2 \delta_{\alpha\beta} - x_\alpha x_\beta\right) d^3 r = J_{\beta\alpha}}$$

simetričen tenzor

„tenzor" je matrika, ki se ustrezno transformira pri rotaciji sistema in je v novem sistemu zveza med količinami enaka. Npr.:

$$\vec L' = R \vec L \quad \text{in} \quad \vec\omega' = R\vec\omega$$

$$\vec L = \underline{J}\,\vec\omega \Rightarrow \vec L' = \underline{J}'\vec\omega' \quad \& \quad \boxed{\underline{J}' = R\,\underline{J}\,R^{-1}}$$

dokaz:

$$\vec L' = R\vec L = R\underline{J}\vec\omega = R\underline{J}R^{-1}R\vec\omega = \underbrace{R\underline{J}R^{-1}}_{\underline{J}'}\vec\omega' .$$

$\underline{J}$ ima 3 lastne vektorje in 3 l. vrednosti;

očitno

$$\underline{J}\,\vec e_\alpha = \lambda_\alpha \vec e_\alpha \qquad \lambda_\alpha = J_{1,2,3}; \quad J_{\alpha\alpha} \geq 0$$

če $J_\alpha \neq J_\beta \Rightarrow \vec e_\alpha \cdot \vec e_\beta = 0$, ker je $\underline{J}$ realen in simetričen.

.. Energija vrtečega se telesa, $(V=0)$

$$T = \frac{1}{2}\sum_i m_i v_i^2 = \frac{1}{2}\sum_i m_i \underbrace{(\vec\omega \times \vec r_i)}_{\vec v_i}\cdot \vec v_i =$$

$$= \frac{1}{2}\sum_i m_i\, \vec\omega\cdot \underbrace{(\vec r_i \times \vec v_i)}_{\vec L} = \frac{1}{2}\vec\omega\cdot\vec L =$$

$W_r = \frac{1}{2} J\omega^2$

$$= \boxed{\frac{1}{2}\,\vec\omega\cdot \underline{J}\,\vec\omega}\,.$$

$\vec e_\alpha \cdot \vec e_\beta = \delta_{\alpha\beta}$

V lastnem sistemu $\underline{J}$ (lastnost telesa, ki sam „pripet" na telo) velja

$$\underline{J} = \begin{pmatrix} J_1 & & \\ & J_2 & \\ & & J_3 \end{pmatrix} \quad \text{in} \quad \vec\omega = \sum_{\alpha=1}^{3}\omega_\alpha \vec e_\alpha$$

ni $\parallel \vec\omega$

$$\underline{J}\vec\omega = \sum_{\alpha=1}^{3} J_{\alpha\alpha}\,\omega_\alpha \vec e_\alpha = \vec L \quad \text{v lastnem sistemu}$$

$$\frac{1}{2}\vec\omega\cdot\underline{J}\,\vec\omega = \frac{1}{2}\sum_{\alpha=1}^{3} J_{\alpha\alpha}\,\omega_\alpha^2 \geq 0 \quad \text{za } \forall\, \vec\omega$$

se vidi iz definicije ($x^2+z^2 \geq 0$) itd.

$$\Rightarrow \boxed{J_{\alpha\alpha} \geq 0.}$$

lastne vrednosti ≥ 0.

$\underline{J}$ je pozitivno definitna matrika.

Enačbe gibanja

$$\frac{d\vec L}{dt} = \left(\frac{d\vec L}{dt}\right)_{\text{rel.}} + \vec\omega\times\vec L = \vec M$$

glede na telo

to $\neq$ nujno 0, kot sledi:

Delajmo v lastnem sistemu $\underline{J}$,

$$\vec{L} = \underline{J}\,\vec{\omega} = \sum_{\alpha=1}^{3} J_{\alpha\alpha}\,\omega_\alpha\,\vec{e}_\alpha = \begin{pmatrix} J_1\omega_1 \\ J_2\omega_2 \\ J_3\omega_3 \end{pmatrix}; \qquad J_2 \leftarrow J_{22} \text{ itd.}$$

V vrtečem se sistemu je v gornjem izrazu od časa lahko odvisen le vektor $\vec{\omega}$; očitno, ker $J_{\alpha\alpha}$ so itak konstante in osi so definirane z $\vec{e}_\alpha$. Torej velja

$$\left(\frac{d\vec{L}}{dt}\right)_{vrt.} = \sum_{\alpha=1}^{3} \underbrace{J_{\alpha\alpha}\,\dot{\omega}_\alpha}_{L_\alpha(t)}\,\vec{e}_\alpha$$

?

torej $\neq 0$ glede na lastni sistem (v splošnem)

Velja še

$$\vec{\omega}\times\vec{L} = \left(\underbrace{(J_3-J_2)\,\omega_2\,\omega_3}_{1},\ (J_1-J_3)\,\omega_3\,\omega_1,\ (J_2-J_1)\,\omega_1\,\omega_2\right)$$

ciklično

$$\begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} \times \begin{pmatrix} J_1\omega_1 \\ J_2\omega_2 \\ J_3\omega_3 \end{pmatrix} = \dots$$

torej $1 \to 2 \to 3$

$$\boxed{\begin{aligned} J_1\dot{\omega}_1 - (J_2-J_3)\,\omega_2\,\omega_3 &= M_1, \\ J_2\dot{\omega}_2 - (J_3-J_1)\,\omega_3\,\omega_1 &= M_2, \\ J_3\dot{\omega}_3 - (J_1-J_2)\,\omega_1\,\omega_2 &= M_3. \end{aligned}}$$

nelinearna 1. reda nehomogena

To so Eulerjeve enačbe; v lastnem sistemu $\underline{J}$.

1. red, nelinearne.

Najenostavnejši primer je, ko ni zunanjih navorov, $\vec{M} = 0$, potem

$$\frac{d\vec{\Gamma}}{dt} = 0 \Rightarrow \vec{\Gamma} = \vec{\Gamma}_0 = \text{konst.} \quad (\text{ponovno} \ldots)$$

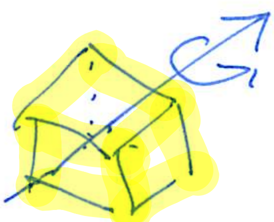
To je zapisano v inercialnem sistemu. V splošnem ne rešimo. Je pa enostavno za $J_1 = J_2$. Če $J_1 = J_2 = J_3 = J$ je trivialno,

$\vec{L} = J\vec{\omega}$

$$J_\alpha \dot{\omega}_\alpha = J \dot{\omega}_\alpha = 0 \quad \text{za } \forall \alpha \Rightarrow \vec{\omega}(t) = \vec{\omega}_0 = \text{konst.}$$

Ponovno pričakujemo, da ko vedno tako, Tako simetrija vendar v splošnem ni. Tako simetrijo ima seveda krogla, pa tudi kocka. Če torej v tudi vrtimo kocko, ko $\vec{\omega} = \text{konst.}$, $\vec{\Gamma} = \text{konst.}$ (seveda) in tudi

$$\vec{\omega} \propto \vec{\Gamma} \; ; \quad \vec{\Gamma} = (J_{ij})\vec{\omega} = J\vec{\omega}.$$



konst.

$\underline{J} = J\,\underline{I}$

Bolj zanimiv je primer, ko

$$J_1 = J_2 \neq J_3$$

še vedno velja

$$J_3 \dot{\omega}_3 = 0 \Rightarrow \omega_3 = \text{konst.} = \omega_0$$

(ne pozabimo: $\vec{e}_3$ je glede na telo) in

$$J_1 \dot{\omega}_1 - (J_1 - J_3)\,\omega_2 \underbrace{\omega_0}_{\text{konst.}} = 0$$
$$J_1 \dot{\omega}_2 - (J_3 - J_1)\,\omega_1 \omega_0 = 0 \qquad /\,:$$

$$\frac{\dot{\omega}_1}{\dot{\omega}_2} = -\frac{\omega_2}{\omega_1} \Rightarrow \left.\begin{aligned}\dot{\omega}_1 &= -\Omega\,\omega_2\\ \dot{\omega}_2 &= \Omega\,\omega_1\end{aligned}\right\} \ddot{\omega}_1 + \Omega^2 \omega_1 = 0$$

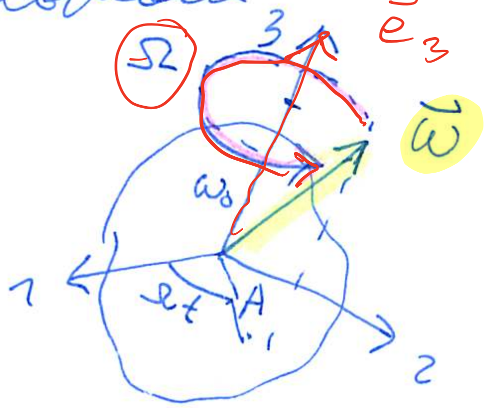
$$\Omega = \frac{J_3 - J_1}{J_1}\,\omega_0$$

očitna rešitev je

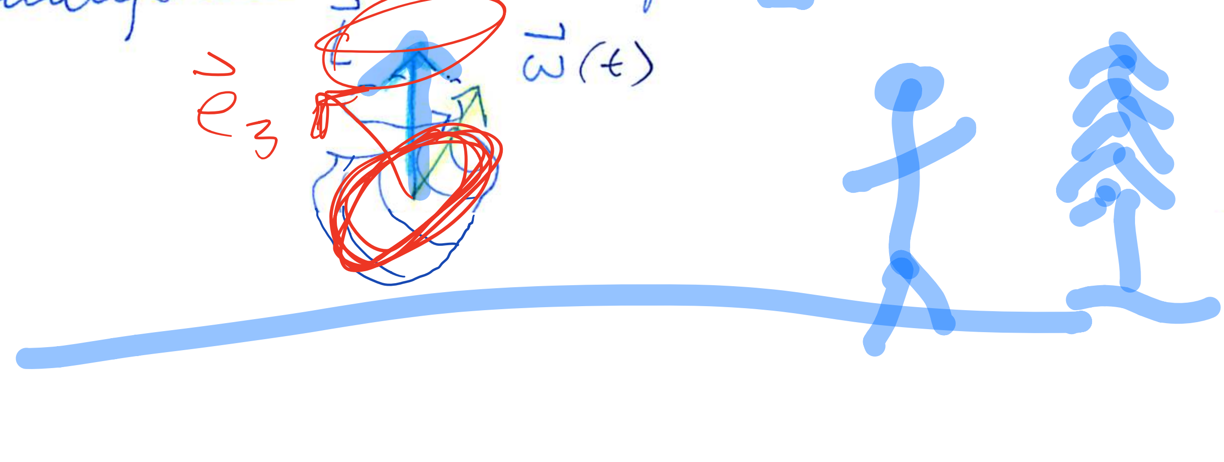
$$\omega_1 = A\cos(\Omega t + \delta) \qquad \omega_1^2 + \omega_2^2 = A^2$$
$$\omega_2 = A\sin(\Omega t + \delta) \qquad |\vec{\omega}|^2 = A^2 + \omega_0^2$$

$$\vec{L} = J\vec{\omega}$$

V lastnem sistemu



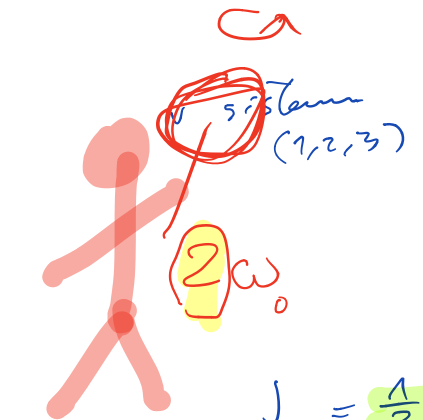
V mirujočem sistemu je $\vec{L}$ = konst.

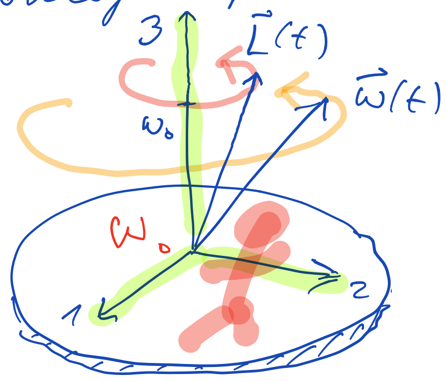


Primeri tresenja (wobble)

Feynman
zgodba za šolite
Mr. F. ?

(a) Frisbee; okrogla ploščica

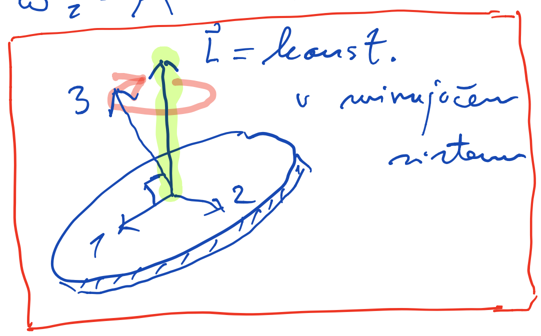




$\omega_3 = \omega_0$

$\omega_1 = A\cos\Omega t$

$\omega_2 = A\sin\Omega t$



$J_3 = \frac{1}{2} m R^2$

$J_1 = J_2 = \ldots = \frac{1}{4} m R^2$

$\left.\begin{array}{l}\omega_0 \\ +\,\omega_0\end{array}\right\}$

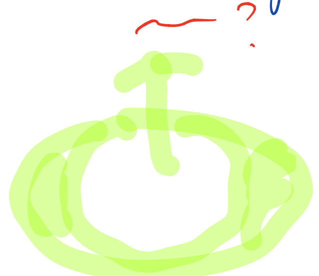
$$\Omega = \frac{J_3 - J_1}{J_1}\,\omega_0 = \frac{\frac{1}{2}-\frac{1}{4}}{\frac{1}{4}}\,\omega_0 = \omega_0 \quad \text{(v lastnem sistemu)}$$

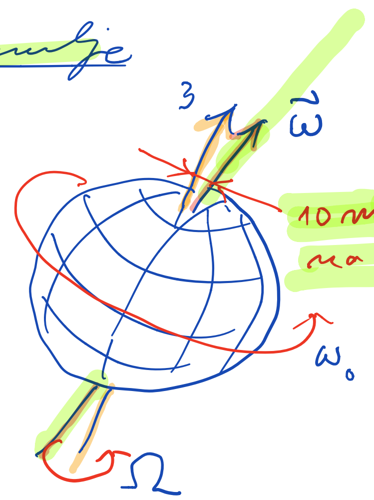
$$\vec{L} = \underline{J}\,\vec{\omega} = \begin{pmatrix} J_1\omega_1 \\ J_2\omega_2 \\ J_3\omega_3 \end{pmatrix}$$

ni $\parallel$ z $\vec{\omega}$ ali z $\vec{e}_3$

opomba: če gledamo od zunaj, je $\Omega = 2\omega_0$
(Feynman v bam...)

(b) tresenje Zemlje
(nutacija).





10 m na polu in se spreminja

$\omega_0 = 2\pi / 1$ dan

$$\frac{J_3 - J_1}{J_1} = 0.00327 \Rightarrow \Omega \sim 2\pi / 300\,\text{dni}$$

v resnici je 433 dni

veliko razlogov za odstopanja
- telesa niso togi
- premiki ledu na polih
- ... ??

31.3.2020

Stabilnost vrtavke

Naj velja sedaj

$$J_1 \neq J_2 \neq J_3 \neq J_1$$

in telo se naj vrti okoli ene od lastnih osi. Zunanjih navorov ni ($\vec{M} = 0$). Eulerjeve enačbe so enake, v lastnem sistemu,

$$J_1 \dot{\omega}_1 - (J_2 - J_3)\omega_2 \omega_3 = 0,$$
$$J_2 \dot{\omega}_2 - (J_3 - J_1)\omega_3 \omega_1 = 0,$$
$$J_3 \dot{\omega}_3 - (J_1 - J_2)\omega_1 \omega_2 = 0.$$

Naj se vrti okoli osi 3,

$$\omega_1 = \omega_2 = 0 \quad \text{in} \quad \omega_3 = \Omega.$$

Kaj se zgodi, če malo spremenimo os vrtenja $\vec{\omega} \to \vec{\omega} + \vec{\eta}$, $|\eta| \ll |\omega|$?

$$\omega_1 = \eta_1,$$
$$\omega_2 = \eta_2,$$
$$\omega_3 = \Omega + \eta_3.$$

vstavimo v E. enačbe

$$J_1 \dot{\eta}_1 - (J_2 - J_3)\eta_2(\Omega + \cancel{\eta_3}) = 0,$$
$$J_2 \dot{\eta}_2 - (J_3 - J_1)(\Omega + \cancel{\eta_3})\eta_1 = 0,$$
$$J_3(\dot{\Omega} + \dot{\eta}_3) - (J_1 - J_3)\cancel{\eta_1 \eta_2} = 0.$$

Ker je $|\eta_i| \ll \Omega$ in $\dot{\Omega} = 0$, velja

$$J_1 \ddot{\eta}_1 = (J_2 - J_3)\,\Omega\,\dot{\eta}_2 + \underbrace{\mathcal{O}(\eta_i \eta_j)}_{\text{zanemarimo}}$$

$$J_2 \dot{\eta}_2 = (J_3 - J_1)\,\Omega\,\eta_1 + \dots$$

$$J_3 \dot{\eta}_3 = 0 + \dots$$

Iz tretje enačbe sledi

$$\eta_3 = \text{konst.}$$

in ko drugo vstavimo v prvo, dobimo

$$J_1 \ddot{\eta}_1 = \frac{(J_2 - J_3)(J_3 - J_1)}{J_2}\,\Omega^2 \eta_1 .$$

torej je

$$C = \pm |C|$$

$$\ddot{\eta}_1 - C\eta_1 = 0, \qquad C = (J_2 - J_3)(J_3 - J_1)\frac{\Omega^2}{J_1 J_2}.$$

$$+|C|$$

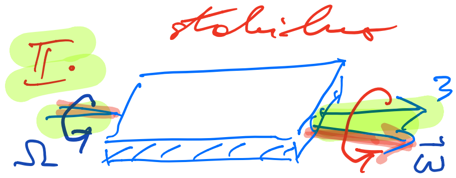
Rešitve:

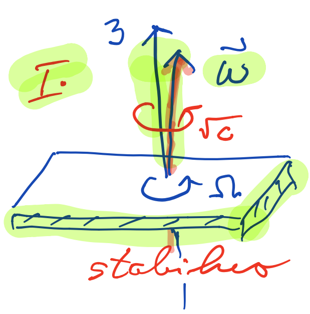
(a) $\boxed{C<0}$: $\eta_1 = A\cos(\sqrt{|C|}\,t + \delta)$ ✓

$$\eta_2 = \frac{J_1}{\Omega(J_2 - J_3)}\dot{\eta}_1 = B\sin(\sqrt{|C|}\,t + \delta) \checkmark$$

$$\frac{J_1}{\Omega(J_2 - J_3)}\,\frac{\Omega}{\sqrt{J_1 J_2}}\sqrt{|(J_2 - J_3)(J_3 - J_1)|} = \sqrt{\frac{J_1}{J_2}\left|\frac{J_3 - J_1}{J_2 - J_3}\right|} = -\frac{B}{A}$$

$\boxed{C<0}$, če $J_2 > J_3$ in $J_3 < J_1$
ali $J_2 < J_3$ in $J_3 > J_1$





$\Longrightarrow$ $J_3 < (J_1, J_2)$ ali $J_3 > (J_1, J_2)$

torej je največji ali najmanjši in gibanje je stabilno in periodično

(b) $\boxed{c > 0}$: $\eta_1 = A_\pm e^{\pm\sqrt{c}\,t}$

$$\eta_2 = B_\pm e^{\pm\sqrt{c}\,t}$$

to je takrat, ko velja

$\rightarrow (J_2 - J_3)(J_3 - J_1) > 0$, oz.

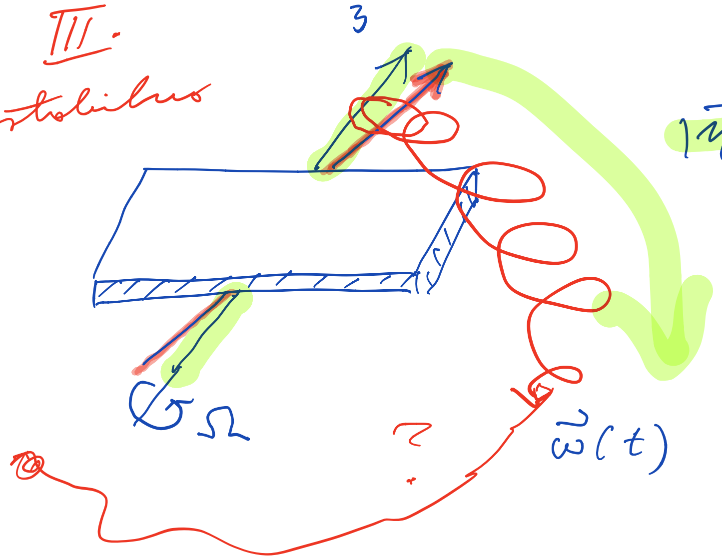
$$J_2 > J_3 \text{ in } J_3 > J_1$$
ali $$J_2 < J_3 \text{ in } J_3 < J_1$$

$$\Rightarrow \boxed{J_1 < J_3 < J_2} \text{ ali } \boxed{J_2 < J_3 < J_1}$$

Torej je med J_1 in J_2 in gibanje je nestabilno

opomba: to velja za $|\vec{\eta}| \ll \vec{\omega}$. Ko se amplituda poveča, $\eta_3 \propto e^{\sqrt{c}\,t}$, linearizirane Eulerjeve enačbe ne veljajo več in se zakomplicira

III. nestabilno



$$|\vec{\eta}| \propto e^{\sqrt{c}\,t}$$

numerična celotna rešitev za $\vec{\eta}(t)$ je periodična (npr. pojav Džanibekova).

Main cause: gravitational torque from the Moon, the Sun and the planets