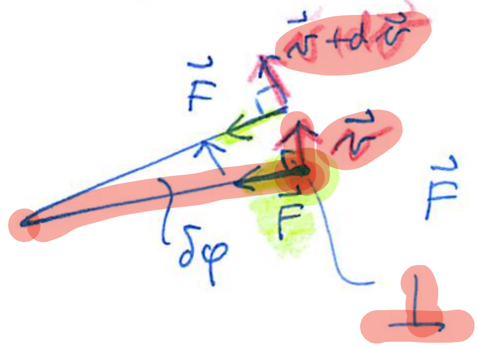
Vrteča osno simetrična vrtavka

Če je vrtavka nesimetrična, ni rešljivo.

Imamo torej $J_1 = J_2 \neq J_3$.

Kaj vemo od fizike 1. letnika?

a) kroženje



$dv = v\,d\varphi$

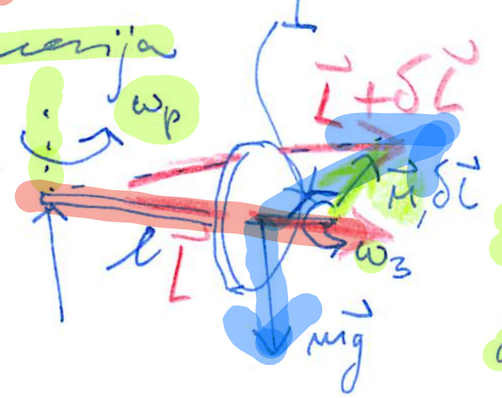
$\vec F \propto d\vec v \perp \vec v$

$$\vec p = m\vec v$$

$$\dot{\vec p} = m\vec a = \vec F$$

$$dp = m\underbrace{v\,d\varphi}_{dv} = F\,dt \Rightarrow \frac{d\varphi}{dt} = \frac{F}{mv} = \frac{F}{p} = \omega$$

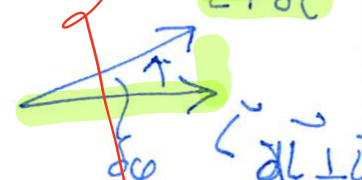
b) precesija



$$\vec L = J\vec\omega_3$$

$$\dot{\vec L} = J\dot{\vec\omega}_3 = \vec M$$

$$dL = J\,d\omega = J\omega\,d\varphi = M\,dt$$



$$\frac{d\varphi}{dt} = \omega_p = \frac{M}{L} = \frac{mgl}{J\omega_3}$$

V obeh primerih je sprememba $\perp$ na količino in velikost konstantna

$\vec F \parallel \delta\vec v \perp \vec v$ in $\vec M \parallel \delta\vec\omega_3 \perp \vec\omega_3$

$\parallel$ $\vec F_c$ $\qquad$ $\parallel$ $mgl\,\frac{\vec M}{M}$

Rezultat: kroženje $\vec p$ oz. $\vec L$, velikost konstantna

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Formalna obravnava vrtavke

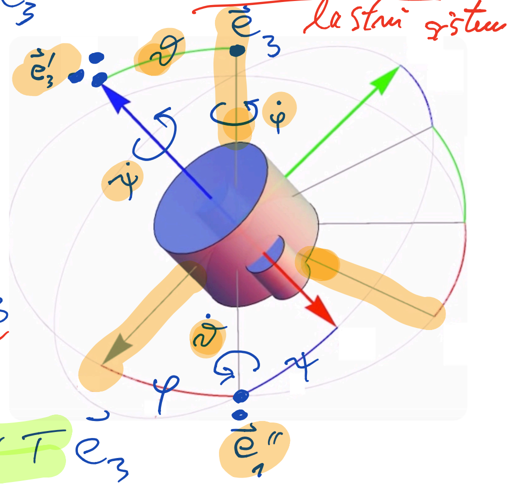
eksperiment ...

- $\dot\varphi$ – precesija
- $\dot\vartheta$ – nutacija
- $\dot\psi$ – rotacija (okoli simetrijske osi)

} Euler

Najprej izrazimo $\boxed{\vec\omega = \dot\varphi\,\vec e_3 + \dot\vartheta\,\vec e_1'' + \dot\psi\,\vec e_3'}$ $\quad \vec e_1', \vec e_2', \vec e_3'$

Vektorje je določiti: $\vec e_3$, $\vec e_1''$ in $\vec e_3'$ v sistemu $(')$. lastni sistem



• $\vec e_3 = \begin{pmatrix}0\\0\\1\end{pmatrix}$ v bazi (x,y,z)

$$\begin{pmatrix}\cos\varphi & \sin\varphi & 0\\ -\sin\varphi & \cos\varphi & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\1\end{pmatrix} = T\vec e_3$$

$$\begin{pmatrix}1 & 0 & 0\\ 0 & \cos\vartheta & \sin\vartheta\\ 0 & -\sin\vartheta & \cos\vartheta\end{pmatrix}\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\ \sin\vartheta\\ \cos\vartheta\end{pmatrix} = VT\vec e_3$$

$$\begin{pmatrix}\cos\psi & \sin\psi & 0\\ -\sin\psi & \cos\psi & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}0\\ \sin\vartheta\\ \cos\vartheta\end{pmatrix} = \begin{pmatrix}\sin\vartheta\sin\psi\\ \sin\vartheta\cos\psi\\ \cos\vartheta\end{pmatrix} = UVT\vec e_3$$

v (x', y', z')

•• $\vec e_1'' = \begin{pmatrix}1\\0\\0\end{pmatrix}$ v bazi (x'', y'', z'')

$$\begin{pmatrix}1 & 0 & 0\\ 0 & \cos\vartheta & \sin\vartheta\\ 0 & -\sin\vartheta & \cos\vartheta\end{pmatrix}\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}1\\0\\0\end{pmatrix} = V\vec e_1''$$

$$\begin{pmatrix}\cos\psi & \sin\psi & 0\\ -\sin\psi & \cos\psi & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}\cos\psi\\ -\sin\psi\\ 0\end{pmatrix} = UV\vec e_1''$$

v (x', y', z')

∴ $\vec e_3' = \begin{pmatrix}0\\0\\1\end{pmatrix}$ v (x', y', z').

∷ Telo se vrti okoli svoje osi, $\dot\psi\,\vec e_3'$, precesira okoli $\vec e_3$, $\dot\varphi\,\vec e_3$, in nutira okoli $\vec e_1''$, $\dot\vartheta\,\vec e_1''$, zato

$\vec\omega = \dot\varphi\,\vec e_3 + \dot\vartheta\,\vec e_1'' + \dot\psi\,\vec e_3'$ ali po komponentah,

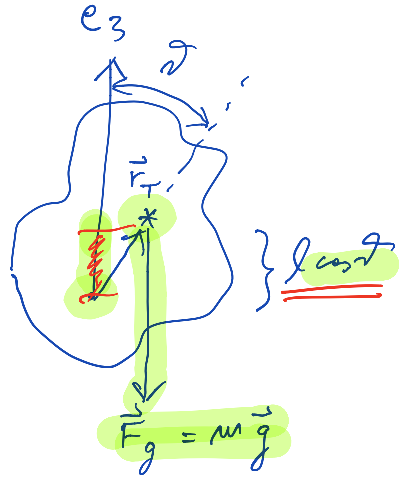
$$\vec\omega(t) = \boxed{\vec\omega = \begin{pmatrix}\sin\vartheta\sin\psi\,\dot\varphi + \cos\psi\,\dot\vartheta\\ \sin\vartheta\cos\psi\,\dot\varphi - \sin\psi\,\dot\vartheta\\ \cos\vartheta\,\dot\varphi + \dot\psi\end{pmatrix}}.$$

v lastnem sistemu

Nadaljujemo v okviru Lagrangeovega formalizma

$$T = \frac{1}{2} J_1 (\omega_1^2 + \omega_2^2) + \frac{1}{2} J_3 \omega_3^2$$

$$V = m g z_T = m g l \cos\vartheta$$



$$\vec{\omega} = \begin{pmatrix} \sin\vartheta \sin\psi\, \dot\varphi + \cos\psi\, \dot\vartheta \\ \sin\vartheta \cos\psi\, \dot\varphi - \sin\psi\, \dot\vartheta \\ \cos\vartheta\, \dot\varphi + \dot\psi \end{pmatrix}$$

$$\omega_1^2 + \omega_2^2 = \sin^2\vartheta \underbrace{(\sin^2\psi + \cos^2\psi)}_{1} \dot\varphi^2 + \underbrace{(\cos^2\psi + \sin^2\psi)}_{1} \dot\vartheta^2 + \underbrace{(\sin\vartheta \sin\psi \cos\psi - \sin\vartheta \cos\psi \sin\psi)}_{0} \dot\varphi \overset{\dot\vartheta}{\cancel{\dot\psi}} =$$

$$= \sin^2\vartheta\, \dot\varphi^2 + \dot\vartheta^2 .$$

$$\omega_3^2 = (\dot\psi + \cos\vartheta\, \dot\varphi)^2 \checkmark$$

torej,

$$L = \frac{1}{2} J_1 (\sin^2\vartheta\, \dot\varphi^2 + \dot\vartheta^2) + \frac{1}{2} J_3 (\dot\psi + \cos\vartheta\, \dot\varphi)^2 - m g l \cos\vartheta .$$

manjkata φ, ψ

$$L = L(\vartheta, \dot\varphi, \dot\vartheta, \dot\psi)$$

↳ kaj so konstante gibanja?

torej:

$$L = \frac{1}{2} J_1 (\sin^2\vartheta\, \dot\varphi^2 + \dot\vartheta^2) + \frac{1}{2} J_3 (\underbrace{\dot\psi + \cos\vartheta\, \dot\varphi}_{\omega_3})^2 - mgl\cos\vartheta.$$

(*)

Opazimo da ψ in φ ciklični koordinati, zato da ustrezna momenta ohranjena:

I. $$p_\psi = \frac{\partial L}{\partial \dot\psi} = J_3 (\underbrace{\dot\psi + \cos\vartheta\, \dot\varphi}_{\omega_3}) = \boxed{J_3\omega_3 \equiv J_1 a} = \text{konst.}$$

od tu je def. a

II. $$p_\varphi = \frac{\partial L}{\partial \dot\varphi} = J_1 \sin^2\vartheta\, \dot\varphi + J_3 \cos\vartheta (\dot\psi + \cos\vartheta\, \dot\varphi) =$$
$$= (J_1 \sin^2\vartheta + J_3\cos^2\vartheta)\dot\varphi + J_3 \cos\vartheta\, \dot\psi \equiv \boxed{J_1 b} = \text{konst.}$$

def. b

III. ohranjena je seveda celotna energija,

$E = T + V$ gl. (*) z +

Sedaj iz I. in II. izrazimo $\dot\psi$ in $\dot\varphi$,

$$J_3 \dot\psi = J_1 a - J_3 \cos\vartheta\, \dot\varphi$$

in to vstavimo v II.,

$$(J_1 \sin^2\vartheta + \cancel{J_3\cos^2\vartheta})\dot\varphi + \cos\vartheta\, \overbrace{(J_1 a - \cancel{J_3 \cos\vartheta\, \dot\varphi})}^{J_3\dot\psi} = J_1 b$$

$$\boxed{\dot\varphi = \frac{b - a\cos\vartheta}{\sin^2\vartheta}}$$

$\vartheta(t) \to \varphi(t)$

&

$$\boxed{\dot\psi = \frac{J_1}{J_3} a - \frac{b - a\cos\vartheta}{\sin^2\vartheta}\cos\vartheta}$$

In še III.:

$$\text{konst.} = E = \frac{1}{2} J_1 \left(\frac{(b - a\cos\vartheta)^2}{\sin^2\vartheta} + \dot\vartheta^2\right) + mgl\cos\vartheta + \underbrace{\frac{1}{2} J_3 \omega_3^2}_{= \text{konst}}$$
$$= \tilde{E} + \frac{1}{2} J_3 \omega_3^2$$

torej:

$$\boxed{\tilde{E} = \frac{1}{2} J_1 \dot\vartheta^2 + \tilde{V}(\vartheta), \text{ kjer je} \quad \tilde{V}(\vartheta) = \frac{1}{2} \frac{(b - a\cos\vartheta)^2}{\sin^2\vartheta} + mgl\cos\vartheta.}$$

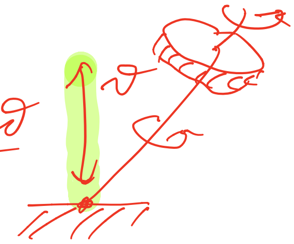
1D osnovna enačba vrtavke $\Rightarrow \vartheta(t)$

Očitno standarden 1-dimenzionalen problem, ki je rešljiv z enim integriranjem. Povsem splošno sicer ne gre, da se pa veliko razumeti:

Velja torej

$$\dot\vartheta^2 = \frac{2}{J_1}\left(\tilde E - \tilde V(\vartheta)\right)$$

$lu = l\cos\vartheta$



$$\frac{d\vartheta}{dt} = \sqrt{\frac{2}{J_1}\left(\tilde E - \tilde V\right)} \quad \text{in}$$

$$t = \sqrt{\frac{J_1}{2}} \int_{\vartheta(0)}^{\vartheta(t)} \frac{d\vartheta}{\sqrt{\tilde E - \tilde V(\vartheta)}}\,,$$ kot že prej nakazano.

Preden gremo računat, si oglejmo obračalne točke. Vemo namreč, da mora veljati $\tilde E \geq \tilde V$, ker $T \geq 0$. V ta namen vpeljemo novo spremenljivko

$u = \cos\vartheta$, $du = -\sin\vartheta\, d\vartheta$, da se znebimo ulomka pod korenom in dobimo polinom v u,

$$t = \int_{u(0)}^{u(t)} \frac{du}{\sqrt{f(u)}}\,,$$

$$P_3(u) = f(u) = \underbrace{(1-u^2)}_{\sin^2\vartheta}\overbrace{(\alpha - \beta u)}^{u^3} - (b - au)^2\,,$$

[da je fizikalno ≥ 0]

$$\boxed{\alpha = \frac{2\tilde E}{J_1}} \quad \text{in} \quad \boxed{\beta = \frac{2mgl}{J_1}}\,; \qquad \dot u^2 = f(u),$$

ker je $\frac{dt}{du} = \frac{1}{\sqrt f} = \frac{1}{|\dot u|}$

$\Longrightarrow u(t)$?

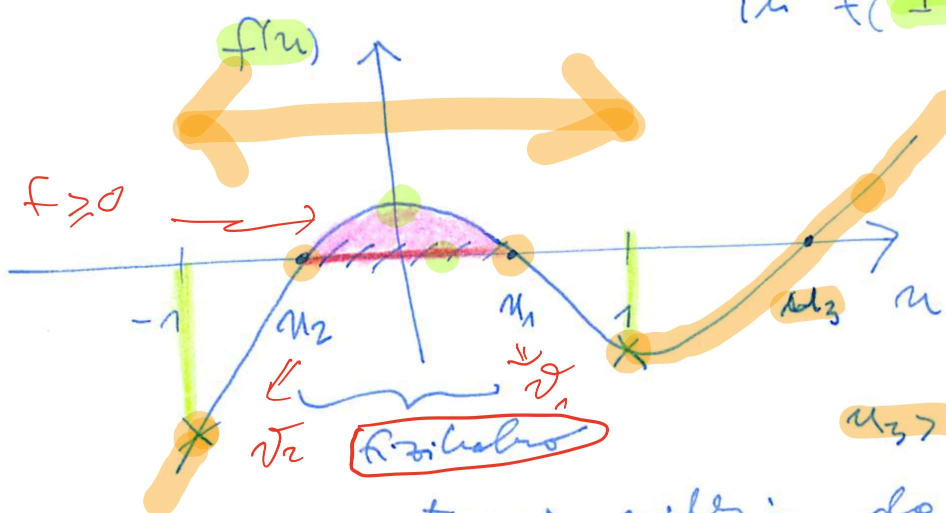
Obračalne točke ($\tilde{E}=\tilde{V}$) so pri $f=0$. Fizikalno dovoljen režim je $f>0$. Zanimajo nas različni primeri α, β, a, b. $f(u)$ je polinom 3. stopnje in integral ni analitično rešljiv splošno.

Limite: $|u|\to\infty$, $u=\cos\vartheta$, $|u|\le 1$:

$f(u) \xrightarrow{|u|\to\infty} \beta u^3$, torej $\pm$ za u $\pm$

in $f(\pm 1) = -(b\mp a)^2 < 0$

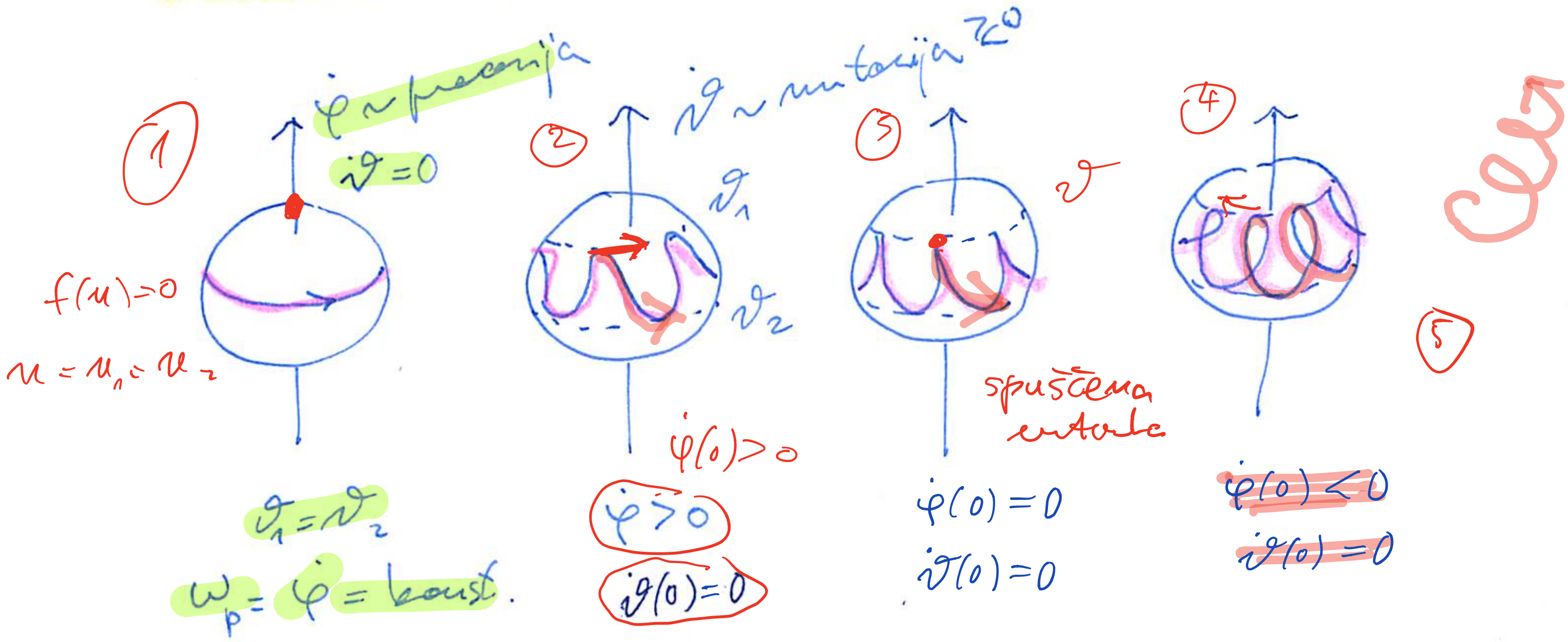
(če $a=b \Rightarrow f(1)=0$.)



$u_3 > 1 \Rightarrow$ ne $\exists\, \vartheta_3$.

Vrtavka se torej giblje, da je $u_2 \le u \le u_1$,

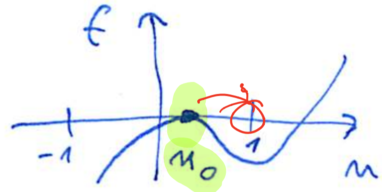
$\vartheta_1 \le \vartheta \le \vartheta_2$.



običajna (enakomerna)
precesija
brez nutacije

V splošnem integrala ne moremo rešiti z osnovnimi funkcijami. Poseben primer ilustrativen:

- Enakomerna precesija



$\dot\vartheta = 0$ in $\dot\varphi = \text{konst.}$, $f(u)$ ima eno samo ničlo,

$$f(u_0) = (1-u_0^2)(\alpha-\beta u_0) - (b-au_0)^2 = 0$$

in

$$\frac{df}{du_0} = 0 = -2u_0(\alpha-\beta u_0) - \beta(1-u_0^2) + 2a(b-au_0)$$

vemo že $\dot\varphi = \dfrac{b-au}{1-u^2}$, $\dot\varphi^2 = \dfrac{(b-au)^2}{(1-u^2)^2}$, zato

$u \to u_0$

$$0 = f(u_0) = \cancel{(1-u_0^2)}(\alpha-\beta u_0) - \frac{(1-u_0^2)^{\cancel{2}}}{(1-u_0^2)}\dot\varphi^2 = 0\,; \quad \text{če } u_0^2 \neq 1$$

$$\alpha - \beta u_0 = \dot\varphi^2(1-u_0^2)$$

$$0 = \frac{df}{du_0} = -2u_0\dot\varphi^2(1-u_0^2) - \beta(1-u_0^2) + 2a\dot\varphi(1-u_0^2)$$

$$\cancel{(1-u_0^2)}\beta = 2a\cancel{(1-u_0^2)}\dot\varphi - 2u_0\dot\varphi^2\cancel{(1-u_0^2)}$$

$$\to \quad \frac{\beta}{2} = a\dot\varphi - \dot\varphi^2 u_0 \qquad \beta = \frac{2}{J_1}mgl$$

p_ψ

$$\frac{mgl}{J_1} = \dot\varphi\left(\frac{J_3\omega_3}{J_1}\right) - \dot\varphi^2\cos\vartheta_0$$

Možni 2 rešitvi; pogoj, da rešitev obstaja,

$$\overset{A}{(-J_1\cos\vartheta_0)}\dot\varphi^2 + \overset{B}{(J_3\omega_3)}\dot\varphi + \overset{C}{(-mgl)} = 0 \Rightarrow \dot\varphi_{1,2} = \frac{-B \pm \sqrt{B^2-4AC}}{2A}$$

$\to \dot\varphi_{1,2} = \ldots$

$$\Rightarrow J_3^2\omega_3^2 \geq 4J_1 mgl\cos\vartheta_0$$

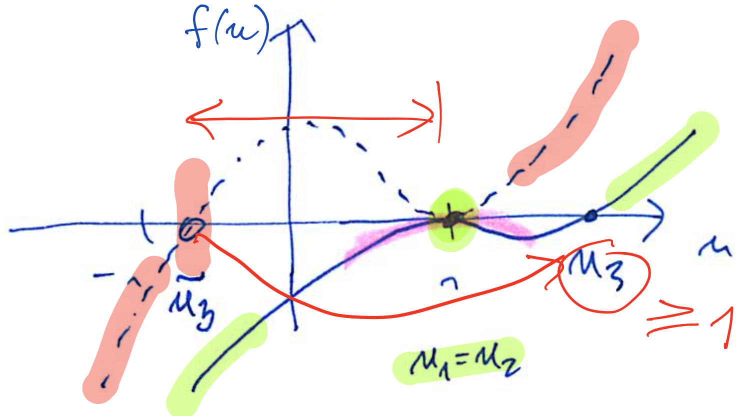
mora zadeti dovolj hitro!

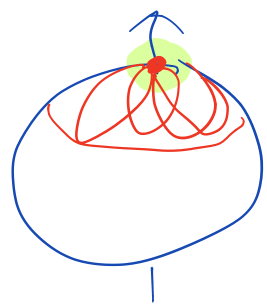
$$\boxed{\omega_3 \geq \frac{2}{J_3}\sqrt{J_1 mgl\cos\vartheta_0}}.$$

hitra vrtavka $\to \ldots$

Speča vrtavka

V bistvu tako, kot prej, samo da imamo edino (dvojno) ničlo pri $u = 1$,





Imamo obe možnosti za $u_3 \gtrless 1$. Fizikalna (stabilna) je samo $u_3 > 1$, ker je potem $u_1 = u_2$ edina možna – stabilna.

Torej, naj bo $\vartheta = 0$ oz. $u = 1$

$\dot\vartheta = 0$ in $f(1) = 0$, in $a = b$, $\alpha = \beta \Rightarrow$

$$f(u) = (1-u^2)(\alpha(1-u)) - a^2(1-u)^2 =$$

$$= \underbrace{(1-u)^2}_{\text{dvojna ničla } u_1 = u_2 = 1}\left(\alpha(1+u) - a^2\right) = 0$$

$u = u_3$: $\quad \alpha(1+u_3) = a^2$

$$u_3 = \frac{a^2}{\alpha} - 1 \geq 1 \Rightarrow \frac{a^2}{\alpha} \geq 2 \;; \quad \alpha = \beta .$$

Torej (glej definicije konstant),

$$\overbrace{\left(\frac{J_3\omega_3}{J_1}\right)^2}^{a^2} \overbrace{\frac{J_1}{2mgl}}^{1/\alpha} \geq \cancel{4}\,2 \quad \text{oz.} \quad \underbrace{\frac{1}{2} J_3 \omega_3^2}_{T} \geq 2\frac{J_1}{J_3} \underbrace{mgl}_{V}$$

$$\boxed{T \geq 2\frac{J_1}{J_3} V}$$

(pogoj, da bo $u_3 \geq 1$, to je, da ni druge rešitve, kot $u_1 = u_2 = 1 \equiv$ speča; ne velja za „cigaro", tj. $J_3 \to 0$).

→ Podrobnosti izpeljave

$$f(u) = (1-u)^2 \left(\alpha(1+u) - a^2 \right) = 0$$

1) $u = 1 = u_1 = u_2$

2) $\alpha(1+u_3) - a^2 = 0 \Rightarrow u_3 = \frac{a^2}{\alpha} - 1$

3) zahtevamo, da je $u_3 \geq 1$,

$$\frac{a^2}{\alpha} - 1 \geq 1 \Rightarrow \frac{a^2}{\alpha} \geq 2$$

$$a = \frac{J_3}{J_1} \omega_3, \quad \alpha = \beta = \frac{2mgl}{J_1}$$

torej je

$$\frac{a^2}{\alpha} = \frac{a^2}{\beta} = \frac{J_3^2 \omega_3^2}{J_1^2} \frac{J_1}{2mgl} \geq 2$$

$$\boxed{\frac{1}{2} J_3 \omega_3^2 \geq 2 \frac{J_1}{J_3} mgl .}$$