Euler-Lagrangeove enačbe za zvezna sredstva

Postopamo analogno diskretnemu primeru, in sicer z upoštevanjem akcije na polici s gostoto ϱ_0,

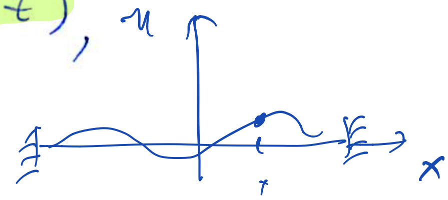
$$S = \int_{t_1}^{t_2} L\, dt = \varrho_0 \int_{t_1}^{t_2}\int_0^X \mathcal{L}\, dx\, dt = \min .$$

gostota police

$L(q, \dot q, t)$

Predpostavimo $\mathcal{L} = \mathcal{L}(u, u_x, u_t, t)$, kar sicer ni nujno. Namesto q $u(x,t)$ vzamemo in jo variiramo,

$$\tilde u(x,t,\alpha) = u(x,t) + \alpha\, \eta(x,t),$$



kjer velja

$$\forall x: \quad \eta(x,t_1) = \eta(x,t_2) = 0 \quad \text{in}$$

$$\forall t: \quad \eta(0,t) = \eta(X,t) = 0.$$

$u_t = \dot u$

novo!

Variiramo:

$$\delta S = \varrho_0 \int_{t_1}^{t_2}\int_0^X \left(\frac{\partial \mathcal{L}}{\partial u}\frac{\partial u}{\partial \alpha} + \frac{\partial \mathcal{L}}{\partial u_t}\frac{\partial u_t}{\partial \alpha} + \frac{\partial \mathcal{L}}{\partial u_x}\frac{\partial u_x}{\partial \alpha} \right) \delta\alpha\, dx\, dt =$$

$$= \delta\alpha\, \varrho_0 \int_{t_1}^{t_2}\int_0^X \left(\frac{\partial \mathcal{L}}{\partial u}\eta + \frac{\partial \mathcal{L}}{\partial u_t}\underbrace{\dot\eta}_{=\frac{\partial \eta}{\partial t}} + \frac{\partial \mathcal{L}}{\partial u_x}\underbrace{\frac{\partial \eta}{\partial x}}_{\eta_x} \right) dx\, dt .$$

Velja

$$\int_{t_1}^{t_2} \frac{\partial \mathcal{L}}{\partial u_t}\frac{\partial \eta}{\partial t}\, dt = -\int_{t_1}^{t_2} \eta \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial u_t}\right) dt + \underbrace{\eta \frac{\partial \mathcal{L}}{\partial u_t}\Big|_{t_1}^{t_2}}_{0},$$

$$\int_0^X \frac{\partial \mathcal{L}}{\partial u_x}\frac{\partial \eta}{\partial x}\, dx = -\int_0^X \eta\frac{d}{dx}\left(\frac{\partial \mathcal{L}}{\partial u_x}\right) dx + \underbrace{\eta \frac{\partial \mathcal{L}}{\partial u_x}\Big|_0^X}_{0} .$$

Torej

$$\delta S = \delta\alpha\, \varrho_0 \int_{t_1}^{t_2}\int_0^X \left[\frac{\partial \mathcal{L}}{\partial u} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial u_t}\right) - \frac{d}{dx}\left(\frac{\partial \mathcal{L}}{\partial u_x}\right) \right] \eta\, dx\, dt = 0$$

za $\forall\ \eta(x,t) \Rightarrow$

$\frac{\partial L}{\partial q}$

$$\boxed{\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial u_t}\right) + \frac{d}{dx}\left(\frac{\partial \mathcal{L}}{\partial u_x}\right) - \frac{\partial \mathcal{L}}{\partial u} = 0.}$$

$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}}$

E.L. en.

x, t

V primeru palice od prej $\mathcal{L} = \frac{1}{2}\varrho u_t^2 - \frac{1}{2} E_Y u_x^2$

„m" „k"

$$\frac{d}{dt}\varrho u_t - \frac{d}{dx} E_Y u_x = \varrho u_{tt} - E_Y u_{xx} = 0. \checkmark$$

valovna enačba!

$$\varrho \frac{\partial^2 u}{\partial t^2} = E_Y \frac{\partial^2 u}{\partial x^2}$$

Kako je z energijo? Poglejmo si še ves slučaj v Hamiltonovem formalizmu. Vpeljemo posplošen impulz,

$$\boxed{\Pi = \frac{\partial \mathcal{L}}{\partial u_t} = \Pi(x,t)}$$

$p = \frac{\partial L}{\partial \dot{q}}$

kanonični spremenljivki sta

torej

$u(x,t)$ in $\Pi(x,t)$ (za palico $= \varrho u_t$) gibalna količina, eno mora v energiji

$p\dot{q} - L = H$

$$H = \varrho_0 \int_0^X (\Pi u_t - \mathcal{L})\, dx = \varrho_0 \int^X \mathcal{H}\, dx,$$

$E = m\dot{x}\dot{x} - \frac{1}{2}m\dot{x}^2 + V(x,t)^0 = H$

$$\mathcal{H} = \frac{1}{2}\varrho u_t^2 + \frac{1}{2}E_Y u_x^2,$$ po pričakovanju

Kdaj se torej energija ohranja?

Pričakujemo, da takrat, ko se nosi na robu ne sklaplja z okolico. Konkretno:

$$0 \overset{?}{=} \frac{dH}{dt} = S_0 \int_0^X \frac{d\mathcal{H}}{dt}\, dx = S_0 \int_0^X (\rho u_t u_{tt} + E_y u_x u_{xt})\, dx$$

valovna enačba od prej → $\rho u_{tt} = E_y u_{xx}$

$u_{xt} = u_{tx}$

torej

$$\frac{dH}{dt} = S_0 E_y \int_0^X (u_t u_{xx} + u_x u_{tx})\, dx =$$

$$= S_0 E_y \int_0^X \frac{d}{dx}(u_t u_x)\, dx = S_0 E_y\, u_t u_x \Big|_0^X .$$

kdaj je: $u_t(0,t)\, u_x(0,t) \overset{?}{=} 0$

ali X X

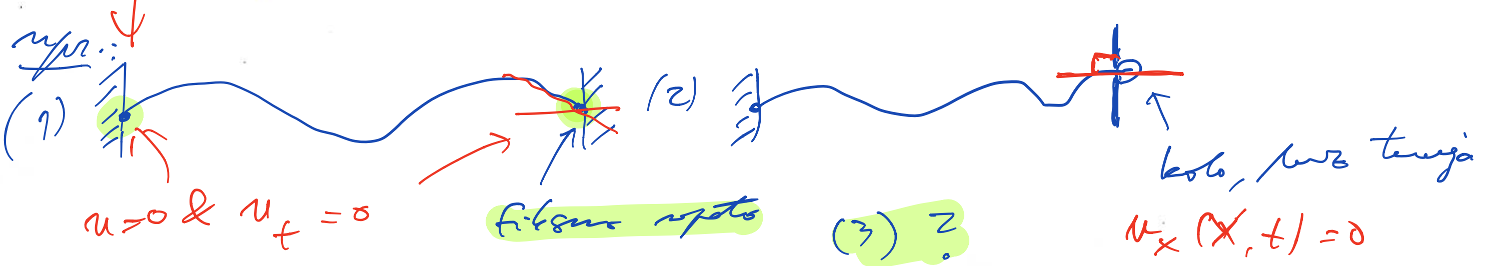
Zagotovo velja, ko za togo vpeto palico (hitrost in odmik na robu $= 0$), so še druge možnosti

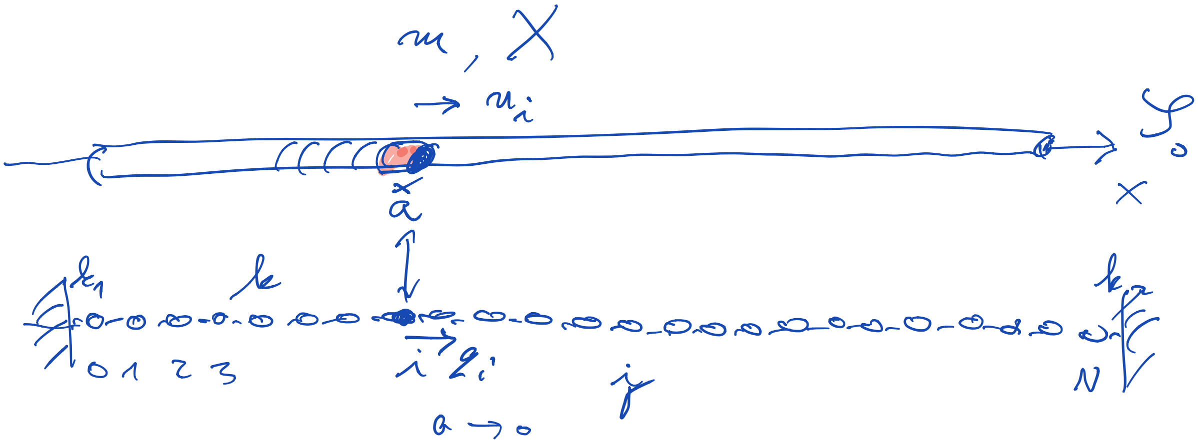
in zato

$$H = E_0 = \text{konst.}$$

→ nehomogena kontinuuma

npr.:





$a \to 0$

$i \to x_i \to x$

$q_i(t)$

$\eta_i = q_i - q_i^0 \to u_i \to u(x_i, t) \to u(x,t)$

$L = \sum_i \frac{1}{2} m_i \dot{q}_i^2 + V(\underline{q})$

$L = \int_0 \mathcal{L}(u, u_x, u_t, t)\,dx$

$u_x^2 \qquad \dot{u}^2 = u_t^2$

!

3D

- $\mathcal{L}(u, u_x, u_y, u_z, u_t, t)$

u skalar

$\underbrace{\frac{\partial u}{\partial x} \quad \frac{\partial u}{\partial y} \quad \frac{\partial u}{\partial z} \quad \frac{\partial u}{\partial t}}_{4}$

.. $\vec{u}$

$\longrightarrow$ teorija polja.

- elektrodinamika .....

- EMP $\quad E, B; (\vec{A}, \varphi) \to A_\mu$

$\mathcal{L}\left(A_\mu, \frac{\partial A_\mu}{\partial x_\nu}, t\right) \longrightarrow$ $(\mu = 1 \dots 4)$

x_μ

valovna enačba

$\mathcal{L}\left(u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial t}, t\right)$ $\qquad u(x,t)$

$\mathcal{L}\left(u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial t}, x, t\right)$

? kdej x ? $\qquad k_i, m_i$

$g(x)$; $\quad E_y(x)$

$E_y(x,t)$