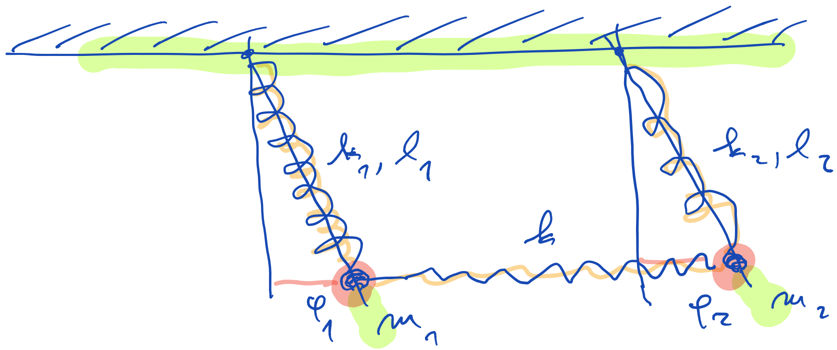
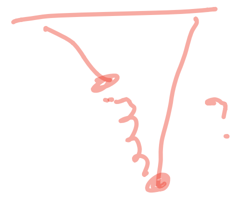
Primer (nvaja)





$$T = \frac{1}{2} m_1 (\dot{l}^2 + l_1^2 \dot{\varphi}_1^2) + \frac{1}{2} m_2 (\dot{l}_2^2 + l_2^2 \dot{\varphi}_2^2),$$

$$+ \frac{1}{2} k_1 (l_1 - l)^2 + \frac{1}{2} k_2 (l_2 - l)^2$$

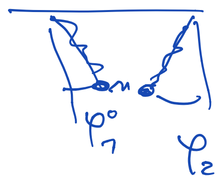
$$V = m_1 g l_1 (1 - \cos\varphi_1) + m_2 g l_2 (1 - \cos\varphi_2)$$

$\frac{1}{2}\varphi_1^2$ $\quad \frac{1}{2}\varphi_2^2$

$$+ \frac{1}{2} k (l_1 \sin\varphi_1 - l_2 \sin\varphi_2)^2.$$

φ_1 $\quad \varphi_2$ $\quad l+\eta_1$ $\quad l+\eta_2$

(če približek)



$q_1 = l_1$ $\qquad \eta_1 = l_1 - l$ $\qquad \dot{l}_1 = \dot{\eta}_1$

$q_2 = l_2$ $\qquad \eta_2 = l_2 - l$ $\qquad \dot{l}_2 = \dot{\eta}_2$

$q_3 = \varphi_1$ $\qquad \eta_3 = \varphi_1 - \varphi_0$? $\qquad \dot{\varphi}_{1,2} = \dot{\eta}_{3,4}$

$q_4 = \varphi_2$ $\qquad \eta_4 = \varphi_2$

$\dot{\eta}\ldots$

če $|\eta_{1,2}| \ll l_{1,2}$

$O(\eta^2)$ $\quad O(\eta^3)$ $\quad O(\eta^4)$

$$l_{1,2}^2 \dot{\varphi}_{1,2}^2 = (l + \eta_{1,2})^2 \dot{\eta}_{3,4}^2 = \boxed{l^2 \dot{\eta}_{3,4}^2} + \cancel{2\eta_{1,2}\dot{\eta}_{3,4}^2 l} + \cancel{\eta_{1,2}^2 \dot{\eta}_{3,4}^2}$$

...

$$V = \frac{1}{2} m_1 g \varphi_1^2 + \frac{1}{2} m_2 g \varphi_2^2 + \frac{1}{2} k l (\varphi_1 - \varphi_2)^2 +$$

višji redi v η_i

$L = \frac{1}{2} \ldots$ $\qquad$?? $\qquad n = 4$

Majhna nihanja

$$\left(l_1^2 \dot\varphi_1 \dot\varphi_1 \quad q_1^2 \dot q_1^2 \right)$$

$$L = T - V = \frac{1}{2} \sum_{ij} \underbrace{m_{ij}(\underline{q})\, \dot q_i \dot q_j}_{m_k v_k^2} - V(\underline{q}) \text{ in}$$

$m_{ij} = m_{ji}$.

Ohranja se energija

$$E = \sum_j \frac{\partial L}{\partial \dot q_j} \dot q_j - L = T + V = H.$$

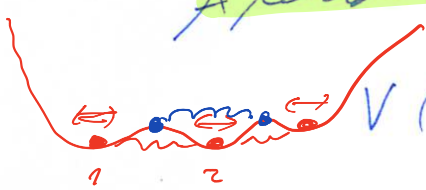
Najprej poiščemo stabilne lege (če so) in razvijemo. Naj bo

$q_i(t) = q_i^0$ in $\dot q_i(t) = 0$ stabilna lega)

$$\left.\frac{\partial L}{\partial q_i}\right|_{\underline{q}^0} = -\left.\frac{\partial V}{\partial q_i}\right|_{\underline{q}^0} = 0 \; ; \quad \underline{q}^0 = (q_1^0, q_2^0, \dots, q_n^0).$$

(sila na malo telo = 0)

Absolutno stabilno stanje, če



$$V(\underline{q}) \geq V(\underline{q}^0) \quad \text{za } \forall\, \underline{q}.$$

Zadošča lokalna stabilnost, to je, če je $\underline{q}$ v okolici $\underline{q}^0$.

Naj bo $V(\underline{q})$ analitična funkcija koordinat

trenje repr.: $F_t = -k F_N \,\mathrm{sign}\,\dot{x}$

$V(\underline{\eta} + \underline{q}^\circ) \quad \propto \; \dot{\eta}_i \dot{\eta}_j$

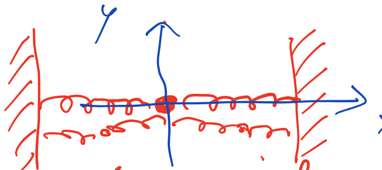
$V \propto x^2$

$$V(\underline{q}) = V(\underline{q}^\circ) + \cancel{\sum_i \frac{\partial V}{\partial q_i}\Big|_{\underline{q}^\circ}} \eta_i + \frac{1}{2} \sum_{ij} \frac{\partial^2 V}{\partial q_i \partial q_j}\Big|_{\underline{q}^\circ} \boxed{\eta_i \eta_j} + \dots$$

$\underbrace{\qquad\qquad}_{0}$ (stacionarnost)

D.N. /

$\frac{\partial U}{\partial x} \sim kx$



$\frac{\partial V}{\partial y} = ?$ in $-k'(y)$, če sta vzmeti prednapeti

$\propto y^3$; če vzmeti nista napeti

$V \propto y^4$

$$\eta_i = q_i - q_i^\circ \quad \text{za } i = 1, \dots n.$$

Torej
$$\frac{1}{2} \sum_{ij} \frac{\partial^2 V}{\partial q_i \partial q_j}\Big|_{\underline{q}^\circ} \eta_i \eta_j > 0 \quad \text{za } \forall\, \eta_i \eta_j$$

- Matrika $\boxed{V_{ij} \overset{\text{def.}}{=} 1 \frac{\partial^2 V}{\partial q_i \partial q_j}\Big|_{\underline{q}^\circ}}$ $= V_{ji}$ je torej pozitivno definitna

- V L je najnižji red v odvodih še

$$w_{ij}(\underline{q}) = w_{ij}(\underline{q}^\circ) + \cancel{\sum_l \frac{\partial w_{ij}}{\partial q_l}\Big|_{\underline{q}^\circ}} \eta_l + \dots \qquad O(\eta^3)$$

$$(l + \eta)\dot{\eta}^2 = l\dot{\eta}^2 + \eta\dot{\eta}^2 \quad ✓$$

to so višji redi: $\underbrace{\eta_l \dot{\eta}_i \dot{\eta}_j}$

Torej je najnižji red kar

$$\boxed{T_{ij} = w_{ij}(\underline{q}^\circ)} \quad \text{in} \quad T_{ij} = T_{ji} \quad \text{in}$$

$$\boxed{\tilde{L} \overset{\text{def.}}{=} \frac{1}{2} \sum_{ij} \left(T_{ij} \dot{\eta}_i \dot{\eta}_j - V_{ij} \eta_i \eta_j \right)} \; ; \qquad L = T - V$$

① ②

③ $\tilde{L} = T - (V - V_0)$,

in
$$\boxed{\sum_j T_{ij} \ddot{\eta}_j + \sum_j V_{ij} \eta_j = 0} \quad \text{za } \forall\, i,$$

$n=1$ $\boxed{T_{11}\ddot{\eta}_1 + V_{11}\eta_1 = 0}$

kar sledi iz

$2\cdot\frac{1}{2}\sum_j T_{ij}\ddot{\eta}_j + 2\frac{1}{2}\sum_j V_{ij}\eta_j = 0$

$$\frac{d}{dt}\frac{\partial \tilde{L}}{\partial \dot{\eta}_i} - \frac{\partial \tilde{L}}{\partial \eta_i} = 0 \quad , \quad \text{za} \quad i = 1, \dots n .$$

Definiramo stolpce,

$$\underline{\eta} = \begin{pmatrix} \eta_1 \\ \eta_2 \\ \vdots \\ \eta_n \end{pmatrix} \quad \text{in} \quad \underline{\eta}^T = (\eta_1, \eta_2, \dots \eta_n)$$

in

$$T_{ij} \to \underline{\underline{T}} = \begin{pmatrix} T_{11} & T_{12} & \dots & T_{1n} \\ \vdots & & & \\ T_{n1} & \dots & & T_{nn} \end{pmatrix}$$

in zato $\left(\frac{1}{2} m\dot{x}^2 - \frac{1}{2} k x^2\right)$

$$\tilde{L} = \frac{1}{2}\left(\dot{\underline{\eta}}^T \underline{\underline{T}}\,\dot{\underline{\eta}} - \underline{\eta}^T \underline{\underline{V}}\,\underline{\eta}\right) ,$$

$\sum_{ij}$

$$\text{in} \quad \underline{\underline{V}} = \begin{pmatrix} V_{11} & V_{12} & \dots & V_{1n} \\ V_{21} & \dots & & \\ \vdots & & & \\ V_{n1} & \dots & & V_{nn} \end{pmatrix} .$$

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opomba: k-ti vektor: $\underline{a}_k = \begin{pmatrix} (a_k)_1 \\ (a_k)_2 \\ \vdots \\ (a_k)_n \end{pmatrix}$ $\qquad \underline{a}_\ell^T = (a_{\ell 1}^T, a_{\ell 2}^T, a_{\ell 3}^T, \dots a_{\ell n}^T)$

$$(\underline{A}\,\underline{a}_k)_i = \sum_j A_{ij}(\underline{a}_k)_j \quad \text{in} \quad (\underline{a}_\ell^T \underline{A})_i = \sum_j (\underline{a}_\ell^T)_j A_{ji}$$

i-ti člen stolpca (element) $\qquad$ i-ti člen v vrstici

Notacija: Vektorji, matrike pri mojih nikovjih

- Vektorje označujemo pod črtano, stolpce:

$$\underline{a} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}.$$

Transponirane so vrstice,

$$\underline{a}^T = (a_1, a_2, \dots a_n).$$

- • Matrike dvakrat pod črtano,

$$\underline{\underline{A}} = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & & & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{pmatrix}$$

in še transponirane,

$$\underline{\underline{A}}^T = \begin{pmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ \vdots & & & \vdots \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{pmatrix}.$$

- • • Delovanje matrike na vektor (množenje):

$$(\underline{\underline{A}}\,\underline{a})_i = \sum_{j=1}^{n} \underline{\underline{A}}_{ij}\,\underline{a}_j \;;$$ rezultat je stolpec.

in še

$$\underline{a}^T \underline{\underline{A}} = (a_1, a_2, \dots a_n) \begin{pmatrix} A_{11} & \cdots & A_{n1} \\ \vdots & & \vdots \\ A_{n1} & \cdots & A_{nn} \end{pmatrix} = \sum_{j=1}^{n} \underline{a}^T_j \underline{\underline{A}}_{ji}$$

rezultat je vrstica.

:: če imamo npr.:

$\underline{a}_k \to$ postopamo analogno: $(\underline{\underline{A}}\,\underline{a}_k)_i = \sum_j \underline{\underline{A}}_{ij} (\underline{a}_k)_j$

itd...

Lastna nihanja

$$\ddot{x} + 2\beta\dot{x} + \omega^2 x = 0 \qquad e^{\pm|\omega|t}$$

$$\eta_i(t) = \alpha\, a_i\, e^{\pm i\omega t} \qquad \omega \text{ je enak za } \forall i$$

$$\ddot{\eta}_i = -\alpha a_i \omega^2 e^{i\omega t}$$

$$\underline{a} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \qquad \sum_j V_{ij} a_j - \omega^2 \sum_j T_{ij} a_j = 0 \quad \text{za } \forall i$$

homogena enačba $\forall \alpha \in \mathbb{C}$

$$\text{oz.} \quad \underline{\underline{V}}\,\underline{a} = \omega^2 \underline{\underline{T}}\,\underline{a} = \lambda \underline{\underline{T}}\,\underline{a} \qquad \boxed{\omega^2 = \lambda \geq 0}$$

$\omega = i|\omega|\,;\ \lambda \leq 0$

Posplošena diagonalizacija; simetričnost modi v $\boxed{\lambda = \omega^2 \in \mathbb{R}}$ in pozitivno definitnost $\omega^2 \geq 0$.

1 parameter

Poseben primer: $\underline{\underline{T}} = T\,\underline{\underline{I}}$ (npr. enake mase)

$T_{ij} \Leftrightarrow w_{ij} \Leftrightarrow m$

$$(\underline{\underline{V}} - \lambda \underline{\underline{T}})\,\underline{a} = 0 \rightarrow (\underline{\underline{V}} - \underbrace{\lambda T}_{\tilde{\lambda}}\,\underline{\underline{I}})\,\underline{a} = 0 \quad (*)$$

$\tilde{\lambda}_k;\ k = 1, 2, \dots n$

$$\text{za } \forall k:\quad (\underline{\underline{V}} - \tilde{\lambda}_k)\,\underline{a}_k = 0 = \underline{\underline{V}}\,\underline{a}_k - \tilde{\lambda}_k \underline{a}_k$$

stolpec

$$\text{in tudi} \quad \underline{a}_\ell^T (\underline{\underline{V}} - \tilde{\lambda}_\ell) = 0$$

vrstica

$$\underline{a}_\ell^T \underline{\underline{V}}\,\underline{a}_k = \tilde{\lambda}_k\, \underline{a}_\ell^T \underline{a}_k$$

$$\underline{a}_k^T \underline{\underline{V}}\,\underline{a}_\ell = \tilde{\lambda}_\ell\, \underline{a}_k^T \underline{a}_\ell$$

$$\underline{a}_\ell^T (\underline{\underline{V}}\,\underline{a}_k) = \underline{a}_\ell^T (\tilde{\lambda}_k \underline{a}_k)$$

$k \quad \ell \quad k \quad \ell\ell$

$$0 = (\tilde{\lambda}_\ell - \tilde{\lambda}_k)\, \underline{a}_\ell^T \underline{a}_k$$

$\tilde{\lambda}_k = \tilde{\lambda}_\ell$ $(\tilde{\lambda}_{\times 4})$ a_k, a_ℓ

$$\Rightarrow \text{če } \tilde{\lambda}_\ell \neq \tilde{\lambda}_k \Rightarrow \boxed{\underline{a}_\ell^T \underline{a}_k = 0} \quad (\delta_{\ell k}) \qquad \underline{a}_\ell^T \underline{a}_k = \delta_{\ell k}$$

Podrobnejši dokazi (po komponentah)

(a)

$$(\underline{\underline{V}} - \tilde{\lambda}_k \underline{\underline{I}})\, \underline{a_k} = 0$$

$$\underline{\underline{V}}\, \underline{a_k} = \tilde{\lambda}_k \underline{a_k}$$

za $\forall i$: $(\underline{\underline{V}}\, \underline{a_k})_i = \sum_j V_{ij}\, \underline{a_k}_j = \tilde{\lambda}_k \underline{a_k}_i$

$$\Rightarrow \underline{a_k}^T \underline{\underline{V}} = \tilde{\lambda}_k \underline{a_k}$$

$$(\underline{a_k}^T \underline{\underline{V}})_i = \sum_j \underline{a_k}_j V_{ji} = \sum_j V_{ij}\, \underline{a_k}_j$$

simetrična

enaki elementi i, samo da je vrstica

(b)

$$\underline{\underline{V}}\, \underline{a_k} = \lambda_k \underline{\underline{T}}\, \underline{a_k} \qquad \text{stolpci}$$

za $\forall i$: $(\underline{\underline{V}}\, \underline{a_k})_i = \sum_j V_{ij}\, \underline{a_k}_j = \lambda_k \sum_j T_{ij}\, \underline{a_k}_j$

$$\Rightarrow \underline{a_k}^T \underline{\underline{V}} = \lambda_k \underline{a_k}^T \underline{\underline{T}} \qquad \text{vrstice}$$

$$\sum_j \underline{a_k}_j \underset{\displaystyle V_{ij}}{\underset{\parallel}{V_{ji}}} = \lambda_k \sum_j \underline{a_k}_j \underset{\displaystyle T_{ij}}{\underset{\parallel}{T_{ji}}}$$

obe matriki simetrični

(c)

$$\underline{a_\ell}^T \underline{\underline{V}}\, \underline{a_k} = \sum_i \underline{a_\ell}_i \sum_j V_{ij}\, \underline{a_k}_j$$

$$\underline{a_k}^T \underline{\underline{V}}\, \underline{a_\ell} = \sum_j \left(\sum_i \underline{a_k}_i V_{ji} \right) \underline{a_\ell}_i$$

enako, ker $V_{ij} = V_{ji}$

Lastne vektorje zložimo v matriko,

$$\underline{\underline{A}} = [\overset{\text{stolpec}}{\underline{a}_1}, \underline{a}_2, \ldots \underline{a}_n] = \begin{array}{c} \\ \begin{bmatrix} a_{11} & \cdots & a_{n1} \\ a_{12} & & a_{n2} \\ \vdots & & \vdots \\ a_{1n} & & a_{nn} \end{bmatrix} \\ \underline{a}_k \end{array}$$

$\underline{\underline{A}}^T$ je transponirana

(*) $$\underline{\underline{A}}^T \underline{\underline{V}} \underline{\underline{A}} - \underline{\underline{A}}^T \underline{\underline{A}} \tilde{\underline{\underline{\Lambda}}} = 0 \; , \quad \tilde{\underline{\underline{\Lambda}}} = \begin{pmatrix} \tilde\lambda_1 & & & \\ & \tilde\lambda_2 & & \\ & & \ddots & \\ & & & \tilde\lambda_n \end{pmatrix}$$

Ker so vektorji $\underline{a}_i$ ortonormirani, je matrika ortogonalna,

$$\underline{\underline{A}}^T \underline{\underline{A}} = \underline{\underline{A}} \underline{\underline{A}}^T = \underline{\underline{I}} \Rightarrow \underline{\underline{A}}^{-1} = \underline{\underline{A}}^T \text{ in}$$

$$\boxed{\underline{\underline{A}}^T \underline{\underline{V}} \underline{\underline{A}} = \tilde{\underline{\underline{\Lambda}}}}$$ diagonaliziranje V-ja.

V praksi imamo bolj zložen primer (kot rečeno, npr. različne mase):

$$\boxed{\underline{\underline{V}}\, \underline{a} = \lambda \underline{\underline{T}}\, \underline{a}}$$ $\underline{\underline{T}}$ ni diagonalna

Podobno, kot prej,

$$\underline{\underline{V}}\, \underline{a}_k = \lambda_k \underline{\underline{T}}\, \underline{a}_k \quad \text{in} \quad \underline{a}_l^T \underline{\underline{V}} = \lambda_l \underline{a}_l^T \underline{\underline{T}}$$

a_l^T $\qquad k = 1, 2, \ldots, n \qquad$ a_k

$$\underline{a}_l^T \underline{\underline{V}}\, \underline{a}_k = \lambda_k\, \underline{a}_l^T \underline{\underline{T}}\, \underline{a}_k$$

$$\underline{a}_k^T \underline{\underline{V}}\, \underline{a}_l = \lambda_l\, \underline{a}_k^T \underline{\underline{T}}\, \underline{a}_l$$

$V_{ij} = V_{ji}$ $\qquad$ $T_{ij} = T_{ji}$

$$0 = (\lambda_k - \lambda_l)\, \underbrace{\underline{a}_l^T \underline{\underline{T}}\, \underline{a}_k}$$

! $\left(\underbrace{\underline{a}_l^T \underline{a}_k}_{\text{prej}} = \delta_{lk} \right)$

če $\lambda_k \neq \lambda_l \Rightarrow \underline{a}_l^T \underline{\underline{T}}\, \underline{a}_k = 0$

izberemo normo: δ_{lk} $\qquad$ ($\underline{\underline{T}}$ pozitivno definitna)

Torej

$$\boxed{\underline{\underline{A}}^T \underline{\underline{T}}\, \underline{\underline{A}} = \underline{\underline{I}}}$$

!

in

$$\boxed{\underline{\underline{A}}^T \underline{\underline{V}}\, \underline{\underline{A}} = \underline{\underline{A}}^T \underbrace{\underline{\underline{T}}\, \underline{\underline{A}}\, \underline{\underline{\Lambda}}} = \underline{\underline{\Lambda}}} = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}$$

λ_k

Normalne koordinate

Imamo torej

$$\tilde{L} = \frac{1}{2} \dot{\underline{\eta}}^T \underline{\underline{T}}\, \dot{\underline{\eta}} - \frac{1}{2} \underline{\eta}^T \underline{\underline{V}}\, \underline{\eta}$$ in

želimo ta izraz diagonalizirati.

Nastavek: $\qquad$ matrika a_{k_i} $\qquad \underline{\underline{A}}$

$$\eta_i(t) = \sum_{k=1}^{n} \alpha_k(t)\, a_{k_i} = \sum_k a_{k_i}\, \alpha_k(t)$$

oz. $\quad \underline{\eta} = \underline{\underline{A}}\, \underline{\alpha}$ in $\underline{\eta}^T = \underline{\alpha}^T \underline{\underline{A}}^T$.

Torej,

$$\tilde{L} = \frac{1}{2}\, \dot{\alpha}^T \underbrace{\underline{\underline{A}}^T \underline{\underline{T}}\, \underline{\underline{A}}}_{\underline{\underline{I}}}\, \dot{\alpha} - \frac{1}{2}\, \alpha^T \underbrace{\underline{\underline{A}}^T V \underline{\underline{A}}}_{\underline{\underline{\Lambda}}}\, \alpha =$$

$$= \frac{1}{2}\, \dot{\alpha}^T \dot{\alpha} - \frac{1}{2}\, \alpha^T \underline{\underline{\Lambda}}\, \alpha, \qquad \underline{\underline{\Lambda}} = \begin{pmatrix} \omega_1^2 & & 0 \\ & \omega_2^2 & \\ 0 & & \ddots\ \omega_n^2 \end{pmatrix}$$

„~~problem~~ je diagonalizacija"

oz.

$$\boxed{\tilde{L} = \frac{1}{2} \sum_k \left(\dot{\alpha}_k^2 - \omega_k^2 \alpha_k^2 \right) \quad \text{in}}$$

ustrezno: $\boxed{\tilde{H} = \frac{1}{2} \sum_k \left(\dot{\alpha}_k^2 + \omega_k^2 \alpha_k^2 \right)}$ in

enačbe gibanja so

$$\boxed{\ddot{\alpha}_k + \omega_k^2 \alpha_k = 0} \quad \text{za } \forall k \ (\text{neodvisni oscilatorji.})$$

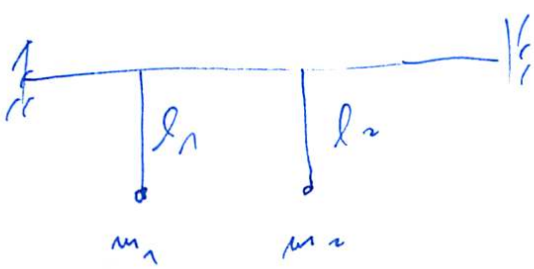
$$\alpha_k = C_k \underset{\cos}{e^{i\omega_k t}} + D_k \underset{\sin}{e^{-i\omega_k t}}$$

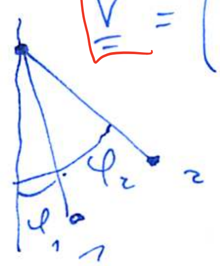
$$\Rightarrow \boxed{\alpha_k(t) = \alpha_{k0} \cos(\omega_k t + \delta_k)}.$$

Primer:

$$\underline{\underline{I}} = \begin{pmatrix} J_1 & 0 \\ 0 & J_2 \end{pmatrix}$$

$$\underline{\underline{V}} = \begin{pmatrix} m_1 g l_1 + D & -2D \\ -2D & m_2 g l_2 + D \end{pmatrix}$$





$$L = \frac{1}{2} J_1 \dot{\varphi}_1^2 + \frac{1}{2} J_2 \dot{\varphi}_2^2 + m_1 g l_1 \cos\varphi_1 + m_2 g l_1 \cos\varphi_2 - \frac{D}{2}(\varphi_1 - \varphi_2)^2$$

$\varphi_1^0 = \varphi_2^0 = 0$, $\eta_i = \varphi_i$

$$\tilde{L} = \frac{1}{2} J_1 \dot{\eta}_1^2 + \frac{1}{2} J_2 \dot{\eta}_2^2 - \frac{m_1 g l_1}{2} \eta_1^2 - \frac{m_2 g l_2}{2} \dot{\eta}_2^2 - \frac{D}{2}(\eta_1 - \eta_2)^2$$