oznake: $\{\ \}$; $[\ ]$; $[\ ]_-$; $[\ ]_+ = \{\ \}$; ....

Poissonovi oklepaji (pišemo zavite!) oglati v QM

Vzemimo poljubno funkcijo koordinat in momentov in časa,

$$f(q_i, p_i, t) = f(\underline{q_i(t)}, \underline{p_i(t)}, t).$$

Poglejmo časovni odvod,

$$\frac{df}{dt} = \sum_i \left(\frac{\partial f}{\partial q_i} \dot{q}_i + \frac{\partial f}{\partial p_i} \dot{p}_i \right) + \frac{\partial f}{\partial t} =$$

$$= \boxed{\sum_i \left(\frac{\partial f}{\partial q_i} \frac{\partial H}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial H}{\partial q_i} \right)} + \frac{\partial f}{\partial t} =$$

$$\stackrel{\text{def.}}{=} \{f, H\} + \frac{\partial f}{\partial t}.$$

; $\frac{df}{dt} = \frac{\partial f}{\partial t} + \{f, H\}$

$H(\underline{q_i}, p_i, t)$

$(\{q_i\}, \{p_i\}, t)$

Definirali smo P. oklepaj,

$$\boxed{\{f, g\} = \sum_i \left(\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right)}.$$

Poissonov

$\frac{\partial f}{\partial q} \frac{\partial g}{\partial p}$

$p \quad q$

Konstanta gibanja,

$f(\vec{r}(t), \vec{p}(t), t)$ $\qquad$ (q, p, t)

$\vec{r}(t)$

$$\boxed{\frac{df}{dt} = \{f, H\} + \frac{\partial f}{\partial t}} \;\nearrow\; = \{f, H\}$$

če $f \neq f(t)$

$\{f, H\} = \{H, f\}$
$= 0$

$\to$ če $\frac{df}{dt} = 0 \Rightarrow \boxed{\{f, H\} = 0}.$

npr. $\{H, H\} = 0$, če ni f. t.

Lastnosti P. oklepajev

$\lambda, \mu \in \mathbb{R};\ \mathbb{C}$

1. linearnost,

$$\{f, \lambda g + \mu h\} = \lambda \{f, g\} + \mu \{f, h\}$$

2. antisimetričnost,

$$\{f, g\} = -\{g, f\}$$

3. produkt,

$f(q, p, t)$

$g(q, p, t)\ ;\ gf = fg$

$h(q, p, t)$

$$\{f, gh\} = \sum_i \left(\frac{\partial f}{\partial q_i} \frac{\partial gh}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial gh}{\partial q_i} \right) =$$

$$= \sum_i \left[\frac{\partial f}{\partial q_i} \left(\frac{\partial g}{\partial p_i} h + g \frac{\partial h}{\partial p_i} \right) - \frac{\partial f}{\partial p_i} \left(\frac{\partial g}{\partial q_i} h + g \frac{\partial h}{\partial q_i} \right) \right] =$$

$$= \{f, g\} h + g \{f, h\}.$$

$$\{f, gh\} = g \{f, h\} + \{f, g\} h.$$

4. Jacobijeva zveza (ciklična),

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0.$$

dokaz: z grobo silo

Primer: Poissonovih oklepajev

$\frac{\partial x}{\partial y} = 0$ $\quad \frac{\partial x}{\partial x} = 1$ $\quad \frac{\partial x}{\partial z} = 0$ $\quad \frac{\partial x}{\partial v_x} = 0 \quad \frac{\partial x}{\partial v_y} = 0$

$\{x, y\} = 0$

$$(a)\quad 0 = \{q_i, q_j\} = \sum_l \Big(\underbrace{\frac{\partial q_i}{\partial q_l}}_{\delta_{il}} \underbrace{\frac{\partial q_j}{\partial p_l}}_{0} - \underbrace{\frac{\partial q_i}{\partial p_l}}_{0} \underbrace{\frac{\partial q_j}{\partial q_l}}_{\delta_{jl}} \Big) = 0$$

enako $0 \rightarrow$

$\{q_i, q_j\} = 0$ in $\{p_i, p_j\} = 0$

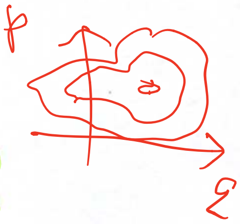
$\{p_x, p_y\} = 0$

$\frac{\partial v_x}{\partial v_x} = 1$, $\quad \frac{\partial v_x}{\partial x} = 0$

vendar

$$\{q_i, p_j\} = \sum_l \Big(\underbrace{\frac{\partial q_i}{\partial q_l}}_{\delta_{il}} \underbrace{\frac{\partial p_j}{\partial p_l}}_{\delta_{jl}} - \underbrace{\frac{\partial q_i}{\partial p_l} \frac{\partial p_j}{\partial q_l}}_{0} \Big) = \delta_{il}\delta_{jl} = \delta_{ij}$$

(SF)



$\{x, p_x\} = 1$

$\{x, p_y\} = 0$

$$\boxed{\{q_i, p_j\} = \delta_{ij}}$$

Osnovna zveza !

(QM)

$i = 1, 2, \ldots n$

q_i in p_i $\overset{\text{def.}}{\Longrightarrow}$ „kanonični spremenljivki" koordinati

$\{q, p\} = 1$

$[x, p] = i\hbar$

- $\dot{q}_i = \{q_i, H\}$
- $\dot{p}_i = \{p_i, H\}$

(b) vrtilna količina, $\vec{L} = \vec{r} \times \vec{p}$:

$x_i \to q_i$

$i = 1,2,3 \leftrightarrow x, y, z$

$$l_x = y p_z - z p_y$$
$$l_y = z p_x - x p_z$$
$$l_z = x p_y - y p_x$$

Računajmo kar zgolj silo:

$$\{l_x, l_y\} = +\frac{\partial l_x}{\partial x}\frac{\partial l_y}{\partial p_x} + \cancel{\frac{\partial l_x}{\partial y}\frac{\partial l_y}{\partial p_y}} + \frac{\partial l_x}{\partial z}\frac{\partial l_y}{\partial p_z} -$$

$1 \to 2 \to$

$$- \cancel{\frac{\partial l_x}{\partial p_x}\frac{\partial l_y}{\partial x}} - \cancel{\frac{\partial l_x}{\partial p_y}\frac{\partial l_y}{\partial y}} - \frac{\partial l_x}{\partial p_z}\frac{\partial l_y}{\partial z} =$$

$$= 0 + p_z 0 + (-p_y)(-x) -$$

$$- 0 - p_z 0 - (y) p_x =$$

$\{l_2, l_1\} = -l_3$

$\{f, f\} = 0$

$$0 \equiv \{l_i, l_i\} = x p_y - y p_x = l_z$$

QM

$[l_i, l_j] = i\hbar\, \varepsilon_{ijk}$

$i(j) = x, y, z$

$$\boxed{\{l_i, l_j\} = \varepsilon_{ijk}\, l_k}$$

i	j	k
1	2	3
2	3	1
3	1	2

→ $i = j \to \varepsilon_{iik} = 0$ 1,2,3 1

$i \leftrightarrow j \to \varepsilon_{ijk} = -\varepsilon_{jik}$ itd.

Zakaj bi to bilo dobro?

- elegantnost izpeljav
- prehod v QM; podobna (ne enaka) struktura.

ali po npr.,

$$\{l^2, l_x\} = \{l_x^2 + l_y^2 + l_z^2, l_x\} =$$

$$= 0 + \{l_y^2, l_x\} + \{l_z^2, l_x\} =$$

$$= 2 l_y \underbrace{\{l_y, l_x\}}_{-l_z} + 2 l_z \underbrace{\{l_z, l_x\}}_{l_y} = 0.$$

$$= 0$$

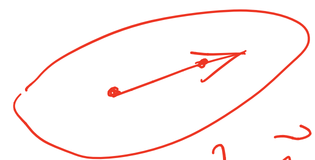
Torej

$$\boxed{\{l^2, l_i\} = 0.}$$

Poisson

c) Laplace–Runge–Lenz D.N.

$$\vec{A} = \vec{p} \times \vec{L} - km \frac{\vec{r}}{r}$$



$|\vec{A}| \sim \varepsilon$

direkten račun pokaže

$$\{l_i, A_j\} = \varepsilon_{ijk} A_k .$$

Vemo že od prej, da če

$$H = \frac{p^2}{2m} - \frac{km}{r} \leftarrow V(r)$$

D.N.

$$\Rightarrow \{\vec{A}, H\} = 0, \quad \{\vec{l}, H\} = 0,$$

$$\{l^2, H\} = 0$$

D.N.: neobvezno

d) $\{l_i, q_j\} = \varepsilon_{ijk} q_k$

$\{l_i, p_j\} = \varepsilon_{ijk} p_k$

e) $\{\vec{p}, \vec{n} \cdot \vec{l}\} = \vec{n} \times \vec{p}$

Kanonične transformacije

Recimo, da zamenjamo spremenljivke,

$q_i \to Q_i(\underline{q}, \underline{p}, t)$ in

$p_i \to P_i(\underline{q}, \underline{p}, t)$.

Transformacija med koordinatami je **kanonična** (def.), če ohranja Poissonove oklepaje,

$$\{f, g\}_{QP} = \{f, g\}_{qp}$$

$$f(\underline{q}(\underline{Q}, \underline{P}), \underline{p}(\underline{Q}, \underline{P}), t) \equiv \tilde{f}(\underline{Q}, \underline{P}, t) \qquad f(\underline{q}, \underline{p}, t)$$

Če je tako, potem so H. enačbe invariantne na transformacijo v smislu

$$\boxed{\dot{Q}_i = \{Q_i, H\}_{pq} + \frac{\partial Q_i}{\partial t} = \{Q_i, H\}_{QP} + \frac{\partial Q_i}{\partial t} = \frac{\partial H}{\partial P_i}} + \underbrace{\frac{\partial Q_i}{\partial t}}$$

$$\boxed{\dot{P}_i = \{P_i, H\}_{qp} + \frac{\partial P_i}{\partial t} = \dots = -\frac{\partial H}{\partial Q_i}} + \underbrace{\frac{\partial P_i}{\partial t}}$$

Če $\frac{\partial Q_i}{\partial t} = 0$ in $\frac{\partial P_i}{\partial t} = 0$, so enačbe ~~enake~~ enake. 0?

Kako se poišče nove koordinate??

Nekateri def. kanoničnost s pogojem $\{Q_i, P_j\} = \delta_{ij}$

Primer

Recimo, da imamo zvezo

$$Q_i = Q_i(\underline{q}, t) \quad \text{in} \quad q_i = q_i(\underline{Q}, t)$$

L zapišimo tako: $L(\underline{q}(\underline{Q}, t), \underline{\dot{q}}(Q, \dot{Q}, t), t)$.

Kako najdemo P_i?

Vemo, da lahko L-u prištejemo popolni odvod,

$$\tilde{L} = L - \frac{dF}{dt}.$$

išČemo momente, iščemo Hamiltonovo f.

Želimo $(\underline{Q}, \underline{P})$, $K(\underline{Q}, \underline{P}, t)$ in da bo

$$\dot{Q}_i = \frac{\partial K}{\partial P_i} \quad \text{in} \quad \dot{P}_i = -\frac{\partial K}{\partial Q_i},$$

torej

$$\tilde{L} = \sum_i \dot{Q}_i P_i - K = \sum_i \dot{q}_i p_i - H - \frac{dF}{dt}.$$

- Izberimo poseben primer, $F = F_1(q, Q, t)$ in

$$\frac{dF_1}{dt} = \sum_i \left(\frac{\partial F_1}{\partial q_i} \dot{q}_i + \frac{\partial F_1}{\partial Q_i} \dot{Q}_i \right) + \frac{\partial F_1}{\partial t}$$

in vstavimo v gornjo enačbo. Dobimo

zveze:

$$\boxed{P_i = -\frac{\partial F_1}{\partial Q_i}}, \quad \boxed{p_i = +\frac{\partial F_1}{\partial q_i}}, \quad \boxed{K = H + \frac{\partial F_1}{\partial t}}.$$

Tako smo „našli" P_i. Opomba: p_i so izraženi s q_i in Q_i.

v informacijo:

V splošnem imamo 4 možnosti za F:

1) $F = F_1(q, Q, t)$: $p_i = \frac{\partial F_1}{\partial q_i}$, $P_i = -\frac{\partial F_1}{\partial Q_i}$, $K = H + \frac{\partial F_1}{\partial t}$

2) $F_2(q, P, t) = F + QP$: $p_i = \frac{\partial F_2}{\partial q_i}$, $Q_i = \frac{\partial F_2}{\partial P_i}$, $K = H + \frac{\partial F_2}{\partial t}$

3) $F_3(p, Q, t) = F - qp$: $p_i = -\frac{\partial F_3}{\partial p_i}$, $P_i = -\frac{\partial F_3}{\partial Q_i}$, $K = H + \frac{\partial F_3}{\partial t}$

4) $F_4(p, P, t) = F_2 - qp$: $q_i = -\frac{\partial F_4}{\partial p_i}$, $Q_i = \frac{\partial F_4}{\partial P_i}$, $K = H + \frac{\partial F_4}{\partial t}$

Legendrove transformacije.