# Hamiltonov formalizem

namesto hitrosti $\dot{q}_i$ delamo z momenti $p_i$,

$$p_i = \frac{\partial L}{\partial \dot{q}_i}$$

$$\frac{dp_i}{dt} = \dot{p}_i = \frac{\partial L}{\partial q_i} \qquad (L\text{-enačbe})$$

Spomnimo se energije. Enako definiramo novo funkcijo, izraženo s $p_i$,

če $T = \sum_{kl} w_{kl}\dot{q}_k\dot{q}_l$

$$\rightarrow \quad H(q_i, p_i, t) \overset{def}{=} p_i\dot{q}_i - L \quad \Big(\text{oz.} = \underset{=}{\sum_i \frac{\partial L}{\partial \dot{q}_i}\dot{q}_i - L}\Big) = T + V \quad (= E)$$

Velja

$$dH = \sum_i \left(\dot{q}_i\, dp_i + \cancel{p_i\, d\dot{q}_i}\right) - \sum_i \left(\frac{\partial L}{\partial q_i} dq_i + \cancel{\frac{\partial L}{\partial \dot{q}_i}} d\dot{q}_i\right) - \frac{\partial L}{\partial t} dt$$

$$= \sum_i \left(\dot{q}_i\, dp_i - \dot{p}_i\, dq_i\right) - \frac{\partial L}{\partial t} dt$$

oz.

$$dH = \sum_i \left(\frac{\partial H}{\partial q_i} dq_i + \frac{\partial H}{\partial p_i} dp_i\right) + \frac{\partial H}{\partial t} dt$$

$H(q, p, t)$

HAMILTONOVE ENAČBE

$$\Rightarrow \boxed{\dot{q}_i = +\frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}, \quad \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}}$$

velja tudi

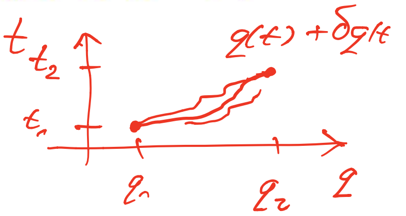
$$\frac{dH}{dt} = \sum_i \Big(\underbrace{\frac{\partial H}{\partial q_i}\overset{\frac{\partial H}{\partial p_i}}{\overset{\parallel}{\dot{q}_i}} + \frac{\partial H}{\partial p_i}\overset{-\frac{\partial H}{\partial q_i}}{\overset{\parallel}{\dot{p}_i}}}_{=0}\Big) + \frac{\partial H}{\partial t} = \frac{\partial H}{\partial t}.$$

# Izpeljava Hamiltonovih enačb iz „Hamiltonovega" principa (minimalna akcija)

Delamo kot pri L, samo da imamo kot spremenljivke $q_i$ in $p_i$ (neodvisne), z akcijo,

min. → $$S = \int_{t_1}^{t_2} L(q_i, \dot q_i, t)\,dt ,$$ $q_i(t)$ ✓

L. enačbe dobimo pri pogoju $\delta q_i(t_1) = \delta q_i(t_2) = 0$ in $\delta S = 0$.



Zapišimo sedaj vse to z $q_i, p_i$,

$$S = \int_{t_1}^{t_2} \left(\sum_i p_i \dot q_i - H\right) dt$$

$$0 = \delta S = \int_{t_1}^{t_2} \sum_i \left(\delta p_i \dot q_i + p_i \delta \dot q_i - \frac{\partial H}{\partial q_i}\delta q_i - \frac{\partial H}{\partial p_i}\delta p_i\right) dt =$$

$$= \int_{t_1}^{t_2} \sum_i \left[\left(\dot q_i - \frac{\partial H}{\partial p_i}\right)\delta p_i + \left(-\dot p_i - \frac{\partial H}{\partial q_i}\right)\delta q_i\right] dt +$$

$$+ \sum_i p_i \delta q_i \Big|_{t_1}^{t_2}$$

per partes

$$\int_{t_1}^{t_2} p_i \delta \dot q_i \, dt = \underbrace{p_i \delta q_i \Big|_{t_1}^{t_2}}_{=0} - \int_{t_1}^{t_2} \dot p_i \delta q_i \, dt$$

i = 1 … n za ∀ $\delta p_i, \delta q_i$ je $\delta S = 0$

$$\Rightarrow \boxed{\dot q_i = \frac{\partial H}{\partial p_i}} \qquad i = 1, 2, \dots n$$

$$\boxed{\dot p_i = -\frac{\partial H}{\partial q_i}} .$$

Upoštevali smo $\delta q_i \Big|_{t_1, t_2} = 0$, za $\delta p_i$ ni takega pogoja

$H(q, p, t)$

(!) ($q_i$ in $p_i$ nista povsem simetrična).

# Primeri Hamiltonovih enačb

1. 1 delec v 1D:

$$T = \frac{1}{2} m \dot{x}^2 , \quad V(x,t)$$

$$L = T - V = \frac{1}{2} m \dot{x}^2 - V(x,t) \; ; \qquad L = L(x, \dot{x}, t)$$

E-L. enačba:

$$\boxed{\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x}} = m\ddot{x} + \frac{\partial V}{\partial x} = 0 \qquad \text{„}F = ma\text{"}$$

$$p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x}$$

$$H = p\dot{x} - L = m\dot{x}^2 - \frac{1}{2} m \dot{x}^2 + V = \frac{1}{2} m \dot{x}^2 + V$$

to še ni $H(x,p,t)$, zato $\dot{x}$ izrazimo s $p$,

$$H(x,p,t) = \boxed{H = \frac{p^2}{2m} + V(x,t)} \; ; \qquad H = T + V$$

H. enačbe

$$\text{I.} \quad \boxed{\dot{x} = \frac{\partial H}{\partial p} = \frac{p}{m}}$$

$$\text{II.} \quad \boxed{\dot{p} = -\frac{\partial H}{\partial x} = -\frac{\partial V}{\partial x}}$$

2 enačbi 1. reda

$$\underset{\ddot{x}}{\underset{\shortparallel}{m a}} = -\frac{\partial U}{\partial x} = F$$

N. e.

$x, p$

$$\dot{x} = \frac{\partial H}{\partial p}$$

$$\dot{p} = -\frac{\partial H}{\partial x}$$

$$\dot{H} = \frac{\partial H}{\partial t} = \frac{\partial V}{\partial t}$$

2. 1 delec v 3D:

$$T = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = \frac{1}{2} m \dot{\vec{r}}^2$$

- $L = \frac{1}{2} m \dot{\vec{r}}^2 - V(\vec{r},t)$

$$m\ddot{\vec{r}} - \nabla V = 0 \; , \qquad \vec{p} = m\dot{\vec{r}}$$

- - $H = T + V = \frac{\vec{p}^2}{2m} + V(\vec{r},t)$

$$\dot{\vec{r}} = \frac{\vec{p}}{m}$$

$$\dot{\vec{p}} = -\nabla V(\vec{r},t)$$

## 3. 1 delec v 2D, polarne koordinate:

$m(q)\dot{q}^2$

- $\mathcal{L} = \frac{1}{2} m(\dot{r}^2 + r^2\dot{\varphi}^2) - V(r,\varphi,t)$

$p_r = m\dot{r} \leftarrow$

$p_\varphi = m r^2 \dot{\varphi}$

- $H = p_r\dot{r} + p_\varphi\dot{\varphi} - \mathcal{L} = m\dot{r}^2 + m r^2\dot{\varphi}^2 - \frac{1}{2} m(\dot{r}^2 + r^2\dot{\varphi}^2) + V =$

$= \frac{1}{2} m(\dot{r}^2 + r^2\dot{\varphi}^2) + V =$

$= \frac{p_r^2}{2m} + \underbrace{\frac{p_\varphi^2}{2mr^2} + V(r,\varphi,t)}_{V_{ef}} = T + V.$

$r$, $p_r$

$$\begin{cases} \boxed{\dot{r} = \frac{p_r}{m}} \\ \boxed{\dot{p}_r} = -\frac{\partial V}{\partial r} \; \textcircled{+} \; \underbrace{\boxed{\frac{p_\varphi^2}{mr^3}}}_{\text{centrifugalna sila}} \end{cases}$$

~ radialni pospešek

$\varphi$, $p_\varphi$

$$\begin{cases} \dot{\varphi} = \frac{p_\varphi}{mr^2} \\ \dot{p}_\varphi = -\frac{\partial V}{\partial \varphi} \end{cases}$$

~ tangentni pospešek (sila).

## 4. N delcev:

statistična fizika

QM

$$H = \sum_{i=1}^{N} \frac{\vec{p}_i^{\,2}}{2m_i} + \sum_{\substack{i,j=1 \\ i<j}}^{N} V(\{\vec{r}_i\}, \{\vec{r}_j\}, t)$$

$$\boxed{\begin{aligned} \dot{\vec{r}}_i &= \frac{\vec{p}_i}{m_i} \\ \dot{\vec{p}}_i &= -\nabla_i \sum_{\substack{ij \\ i<j}} V \end{aligned}} = \dots$$

? $\xrightarrow{\text{če}}$ $V(|\vec{r}_{ij}|)$ …

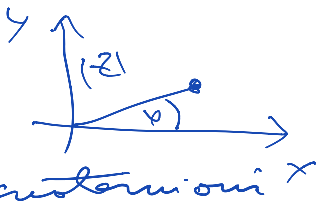
• $\rightarrow$ $V(\vec{r}_{ij})$ …

$\rightarrow$ ?

# Hamilton

$x \in \mathbb{R}$

$z = x + iy \in \mathbb{C}$ $\qquad i^2 = -1$



več dimenzij

v 4 dim: kvaternioni

$q = x + i y + j z + k t \qquad x, y, z, t \in \mathbb{R}$

$i^2 = -1$; $j^2 = -1$; $k^2 = -1$

$ijk = -1$; $ik = -ki$; ...

$i^2 = j^2 = k^2 = ijk = -1$

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Paulijeve matrike

$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\qquad \sigma_x \sigma_y = -\sigma_y \sigma_x$ itd

$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ $\qquad \sigma_x \sigma_y \sigma_z = i$

$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\Rightarrow$ $i\sigma_x$, $i\sigma_y$, $i\sigma_z$ imajo lastnosti kvaternionov