Nabit delec v magnetnem polju: Lagrangeev in Hamiltonov pristop

Že od prej vemo, da je potencial lahko odvisen od hitrosti in je sila podana z

$$F_i = -\frac{\partial U(q_i,\dot q_i,t)}{\partial q_i} + \frac{d}{dt}\frac{\partial U(q_i,\dot q_i,t)}{\partial \dot q_i}\;. \quad \text{(posplošena sila)} \quad (*)$$

Ker je sila na nabit delec v magnetnem polju odvisna od hitrosti, bomo imeli tak primer. Poiščimo $U(\vec r, \dot{\vec r}, t)$!

Velja Lorentzova sila (*)

$$\boxed{\vec F = e(\vec E + \vec v \times \vec B)}.$$

Običajno se nadaljuje z uvedbo skalarnega in vektorskega potenciala, $(\phi, \vec A)$,

4 dim. ${}^4\!A = (\phi, \vec A)$

$$\boxed{\vec B = \nabla \times \vec A} \quad \text{in} \quad \vec E = -\nabla\phi - \frac{\partial \vec A}{\partial t},$$

kar očitno zadošča Maxwellovim enačbam

$\nabla\cdot(\nabla\times\vec v) = 0$ za $\forall \vec v$

✓ $\boxed{\nabla\cdot\vec B = 0}$ in

$$\boxed{\nabla\times\vec E} = \underbrace{-\nabla\times\nabla\phi}_{0} - \nabla\times\frac{\partial\vec A}{\partial t} = -\frac{\partial}{\partial t}\nabla\times\vec A = \boxed{-\frac{\partial \vec B}{\partial t}}\ ✓$$

Silo torej lahko izrazimo takole

$$\boxed{\vec F = e\left[-\nabla\phi - \frac{\partial\vec A}{\partial t} + \vec v\times(\nabla\times\vec A)\right]},$$

$\vec A(\vec r, t)$

$\phi(\vec r, t)$

$q_i, \dot q_i$

$V(\dots) = ?$

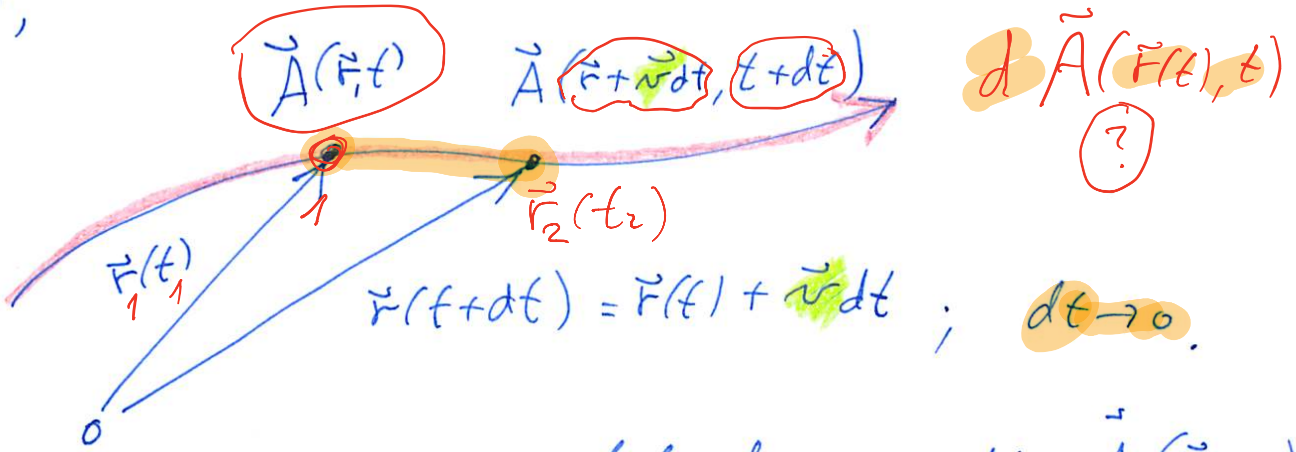
kar bomo sedaj predelali v obliko (*);

(člen $\frac{d}{dt}\frac{\partial U}{\partial \dot q_i}$ je časovni odvod gradienta glede na $\vec v$)

$\frac{d}{dt}\nabla_{\vec v} U$; $\vec F = -\frac{\partial}{\partial \vec r}U + \frac{d}{dt}\frac{\partial U}{\partial \dot{\vec r}}$ (*)

Spomnimo se „substancialnega" odvoda.

Imamo delec, ki se giblje po trajektoriji $\vec r(t)$,



Delec se giblje v vektorskem polju $\vec A(\vec r,t)$ in v časovnem intervalu dt se na mestu delca polje spremeni za $d\vec A$,

$$dA_x = A_x(\vec r+\vec v dt, t+dt) - A_x(\vec r,t) =$$

$$= \frac{\partial A_x}{\partial x}\underset{v_x dt}{\underset{\shortparallel}{dx}} + \frac{\partial A_x}{\partial y}\underset{v_y dt}{\underset{\shortparallel}{dy}} + \frac{\partial A_x}{\partial z}\underset{v_z dt}{\underset{\shortparallel}{dz}} + \frac{\partial A_x}{\partial t}dt =$$

$$= \left[\left(v_x\frac{\partial}{\partial x} + v_y\frac{\partial}{\partial y} + v_z\frac{\partial}{\partial z}\right)A_x + \frac{\partial A_x}{\partial t}\right]dt =$$

gradient skalar

$$= \left(\vec v\cdot\nabla A_x + \frac{\partial A_x}{\partial t}\right)dt$$

Enako velja za y in z, torej

$$\boxed{\frac{d\vec A}{dt} = \frac{\partial \vec A}{\partial t} + \underbrace{(\vec v\cdot\nabla)\vec A}}$$

vektor: $\begin{pmatrix}\vec v\cdot\nabla A_x\\ \vec v\cdot\nabla A_y\\ \vec v\cdot\nabla A_z\end{pmatrix}$ ~ „odvod v smeri $\frac{\vec v}{v}$" $\cdot v$

Velja tudi zveza

$\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c} = \dots$? ∇

$$\vec{v}\times(\nabla\times\vec{A}) = \nabla(\vec{v}\cdot\vec{A}) - (\vec{v}\cdot\nabla)\vec{A},$$

kjer je $\vec{v}$ hitrost delca. Dokažemo z grobo silo po komponentah (ali verjamemo...):

Leva stran (komponenta x):

$$x:\quad v_y(\nabla\times\vec{A})_z - v_z(\nabla\times\vec{A})_y =$$

$$= v_y\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) - v_z\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)$$

Desna stran:

$$\frac{\partial}{\partial x}(v_x A_x + v_y A_y + v_z A_z) - \left(v_x\frac{\partial}{\partial x} + v_y\frac{\partial}{\partial y} + v_z\frac{\partial}{\partial z}\right)A_x =$$

$$= \underbrace{v_x\frac{\partial A_x}{\partial x} - v_x\frac{\partial A_x}{\partial x}}_{=0} + v_y\frac{\partial A_y}{\partial x} - v_y\frac{\partial A_x}{\partial y} + v_z\frac{\partial A_z}{\partial x} - v_z\frac{\partial A_x}{\partial z}.$$

D. N.

✓ res enako

$\to x,\ \to y, z$

Vrnimo se nazaj na

$$\vec{F} = e\left[-\nabla\phi - \frac{\partial\vec{A}}{\partial t} + \vec{v}\times(\nabla\times\vec{A})\right]$$

in uporabimo:

$$\boxed{\frac{d\vec{A}}{dt} = \frac{\partial\vec{A}}{\partial t} + \nabla(\vec{v}\cdot\vec{A}) - \vec{v}\times(\nabla\times\vec{A})},$$

$\nabla(\vec{r}\cdot\vec{A}) = \vec{A}$

$\frac{d}{dt}\vec{A} = \frac{d}{dt}\nabla_{\vec{v}}(\vec{v}\cdot\vec{A})$

dobimo

$(-)$

$$\vec{F} = e\left[-\nabla_{\vec{r}}(\phi - \vec{v}\cdot\vec{A}) - \frac{d\vec{A}}{dt}\right] =$$

$$= e\left[-\nabla_{\vec{r}}(\phi - \vec{v}\cdot\vec{A}) + \frac{d}{dt}\left(-\nabla_{\vec{v}}(\vec{v}\cdot\vec{A})\right)\right] =$$

$$= -\nabla_{\vec{r}}U + \frac{d}{dt}\nabla_{\vec{v}}U,$$

če

$$\boxed{U = e\phi - e\vec{v}\cdot\vec{A}} = U(\vec{r},\vec{v},t).$$

$\vec{F} = -\nabla_{\vec{r}}U + \frac{d}{dt}\nabla_{\vec{v}}U$ ✓

$\frac{\partial U}{\partial \dot{q}_i}$

$\vec{F} = e\vec{E} + e\vec{v}\times\vec{B}$

Lagrangeova funkcija je torej $\vec B = \nabla \times \vec A$??

$$L = \frac{1}{2} m v^2 - e\phi + e\vec v\cdot\vec A$$

in ustrezne enačbe gibanja so izraz za Lorentzovo silo in posplošen moment je

$p_q = \frac{\partial L}{\partial \dot q}$

$$\vec p = \nabla_{\vec v} L = m\vec v + e\vec A .$$

!

Hamiltonova funkcija je definirana (*)

$$E = H = \vec p\cdot\vec v - L = \; ! \; \boxed{H = \frac{|\vec p - e\vec A|^2}{2m} + e\phi} \Rightarrow \vec F = m\vec a$$

$$= m v^2 + e\vec v\cdot\vec A - \frac{1}{2} m v^2 + e\phi - e\vec v\cdot\vec A =$$

$(\phi, \vec A)$

kje je $\vec A$? $\to ?\vec p$

!

$$= \frac{1}{2} m v^2 + e\phi = \leftarrow$$ še ni izraženo z impulzi

$\frac{p^2}{2m} + V$

$$= \frac{|\vec p - e\vec A|^2}{2m} + e\phi = H(q,p,t) \qquad m\vec v = \vec p - e\vec A \; !!$$

ne: $\dot q$

Komentar: Lorentzova sila je $\perp$ na $\vec v$, $e\vec v \times \vec B$, zato je $P = \vec F\cdot\vec v = 0$, torej polje ne deluje z močjo (ne spreminja mehanskega dela). Torej ne spremeni energije in zato ne nastopi v izrazu $\frac{1}{2} m v^2 + e\phi$. $\vec B$ ✗; $\vec E$ ✓

Vpliva pa, seveda, na obliko tira.

q

$\sim e^{-\beta H}$; $\beta = \frac{1}{k_B T}$

Statistična fizika (*)

! $\vec B$!

QM:
$$\hat H = \frac{(\vec p - e\vec A)^2}{2m} + e\phi = \frac{(-i\nabla - e\vec A)^2}{2m} + V,$$

enote: $\hbar = 1$