Primeri Lagrangeovih enačb

① • prost (brez mez) delec v potencialu $V(\vec r)$

$$L = T - V = \frac{1}{2} m v^2 - V(\vec r) =$$

$$= \frac{1}{2} m (\dot x^2 + \dot y^2 + \dot z^2) - V(x, y, z) \; ; \quad q_{1,2,3} = x, y, z$$

E-L. $$\frac{d}{dt}\frac{\partial L}{\partial \dot x} - \frac{\partial L}{\partial x} = \frac{d}{dt}(m\dot x) + \frac{\partial V}{\partial x} = 0 \Rightarrow m\ddot{\vec r} = \underbrace{-\nabla V}_{\vec F} \checkmark$$

y, z (enako)

$\equiv$ N.

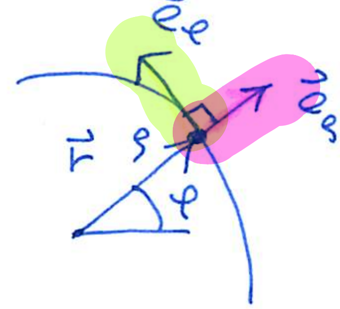
② .. gibanje v ravnini (polarne koordinate)

$$\vec r = \rho(\cos\varphi, \sin\varphi) = \rho\,\hat e_\rho \; ; \quad q_1 = \rho, \; q_2 = \varphi$$

$$\dot{\vec r} = \dot\rho(\cos\varphi, \sin\varphi) + \rho(-\sin\varphi, \cos\varphi)\dot\varphi =$$

$$= \dot\rho\,\hat e_\rho + \rho\dot\varphi\,\hat e_\varphi$$

radialni   tangentni
enotni vektor



$$\frac{\partial \dot r_i}{\partial \dot q_j} = \frac{\partial r_i}{\partial q_j}$$

$$\frac{\partial \dot{\vec r}}{\partial \dot\rho} = \frac{\partial \vec r}{\partial \rho} = \hat e_\rho$$

in

$$\frac{\partial \dot{\vec r}}{\partial \dot\varphi} = \frac{\partial \vec r}{\partial \varphi}, \text{ kot smo od prej}$$

$$\frac{d}{dt}\frac{\partial \vec r}{\partial \rho} = \frac{d}{dt}\hat e_\rho = \dot\varphi\,\hat e_\varphi = \boxed{\frac{\partial \dot{\vec r}}{\partial \rho}}$$

$$\frac{d}{dt}\frac{\partial \vec r}{\partial \varphi} = \frac{d}{dt}\rho\,\hat e_\varphi = \dot\rho\,\hat e_\varphi - \rho\dot\varphi\,\hat e_\rho = \boxed{\frac{\partial \dot{\vec r}}{\partial \varphi}}$$

$$T = \frac{1}{2} m v^2 = \frac{1}{2} m (\dot\rho^2 + \rho^2\dot\varphi^2), \text{ ker } \hat e_\rho \cdot \hat e_\varphi = 0$$

$\dot x^2 + \ldots$

torej:

$$L = \frac{1}{2} m (\dot\varrho^2 + \varrho^2 \dot\varphi^2) - V(\varrho, \varphi)$$

$q_j \; ; \; j = 1,2 \qquad z = m \qquad 3 = N$

$j=1;\ \varrho$:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot\varrho} - \frac{\partial L}{\partial \varrho} = 0$$

$$\frac{d}{dt} m\dot\varrho - \left(m\varrho\dot\varphi^2 - \frac{\partial V}{\partial \varrho}\right) = 0$$

$$\boxed{m\ddot\varrho = m\varrho\dot\varphi^2 - \frac{\partial V}{\partial \varrho}}$$

$j=2;\ \varphi$:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot\varphi} - \frac{\partial L}{\partial \varphi} = 0$$

$$\frac{d}{dt} m\varrho^2\dot\varphi - \left(-\cancel{\frac{\partial V}{\partial \varphi}}\right) = 0 \qquad -\frac{\partial V}{\partial \varphi}$$

$$2m\varrho\dot\varrho\dot\varphi + m\varrho^2\ddot\varphi = -\frac{\partial V}{\partial \varphi}$$

$$\boxed{\frac{d}{dt}(m\varrho^2\dot\varphi) = -\frac{\partial V}{\partial \varphi}} = 0$$

če V ni odvisen od φ, $V = V(r)$, t.j. centralni;

$$\frac{\partial V}{\partial \varphi} = 0 \Rightarrow m\varrho^2\dot\varphi = \text{konst.} = l_0 \qquad \varrho(t),\ \varphi(t)$$

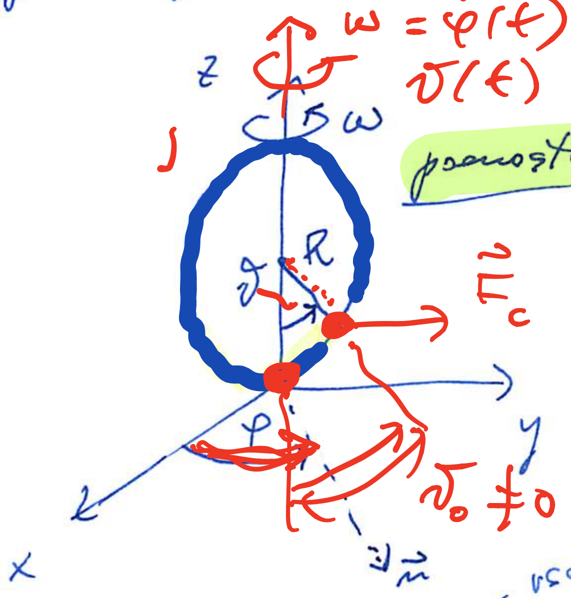
notranja količina sveda.

$$\Rightarrow m\ddot\varrho = m\varrho\frac{l_0^2}{m^2\varrho^4} - \frac{\partial V}{\partial \varrho} \Rightarrow$$

$$m\ddot\varrho = -\frac{\partial}{\partial \varrho}(V(\varrho) + V_c); \quad V_c = \frac{l_0^2}{2m\varrho^2} \quad \text{„centrifugalni" potencial}$$

$$V_{ef}(\varrho) = V(\varrho) + V_c$$

(3) gibanje mase na vrtljivem obroču:



$q_1=\varphi$; $q_2=\vartheta$

poenostavitev: naj bo $\varphi=\omega t$; $\omega=$ konst.

$x = R\sin\vartheta\cos\omega t$; $\dot x = \dots$

$y = R\sin\vartheta\sin\omega t$; $\dot y = \dots$

$z = R - R\cos\vartheta$; $\dot z = \dots$

kvadrat komponent hitrosti

vsota

$$T = \frac{1}{2} m\dot{\vec r}^2 = \frac{1}{2} mR^2\left(\dot\vartheta^2 + \omega^2\sin^2\vartheta\right)$$

$\dot\varphi^2$

$\dot\varphi = \omega =$ konst

$0 \longrightarrow$ $V = mgz = -mgR\cos\vartheta$

$$L = T - V = \frac{1}{2} mR^2\dot\vartheta^2 - V_{ef}$$

$$V_{ef} = \underbrace{-mgR\cos\vartheta}_{z} - \frac{1}{2} mR^2\omega^2\sin^2\vartheta$$

~ centrifugalni

$$\frac{d}{dt}\frac{\partial L}{\partial\dot\vartheta} - \frac{\partial L}{\partial\vartheta} = 0$$

če φ spustimo:

$$\frac{d}{dt}\frac{\partial L}{\partial\dot\varphi} - \frac{\partial L}{\partial\varphi} = 0 = \frac{d}{dt}\left(mR^2\sin^2\vartheta\,\dot\varphi\right)$$

$\Rightarrow \underbrace{(mR^2\sin^2\vartheta)\dot\varphi}_{\text{vrtilna}} =$ konst. količina

$$\frac{d}{dt}\left(mR^2\dot\vartheta\right) + \frac{\partial V_{ef}}{\partial\vartheta} = 0$$

$$mR^2\ddot\vartheta = -\frac{\partial V_{ef}}{\partial\vartheta}$$

stacionarna stanja: $\dot\vartheta = 0$ & $\ddot\vartheta = 0 \Rightarrow \dfrac{\partial V_{ef}}{\partial\vartheta} = 0$

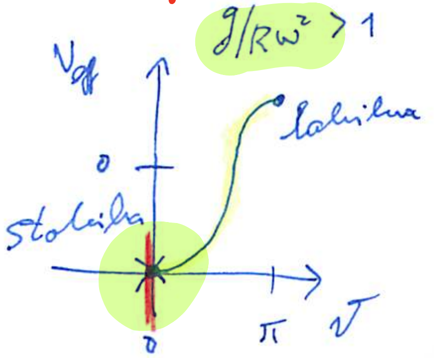
(Σ sil na telo v vrtečem sistemu $=0$)

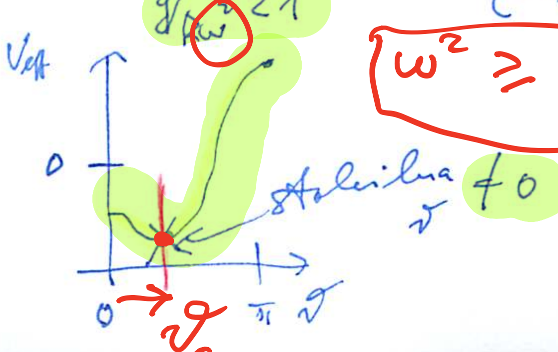
$\dfrac{\partial V}{\partial\vartheta}=0 \Rightarrow g\sin\vartheta = R\omega^2\sin\vartheta\cos\vartheta \Rightarrow$ 1) $\sin\vartheta = 0$ ali

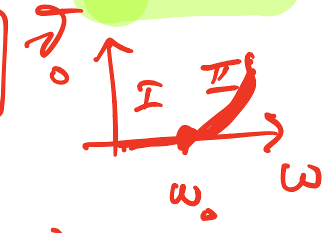
2) $\cos\vartheta = \dfrac{g}{R\omega^2} \leq 1$

$g/R\omega^2 > 1$ $\qquad$ $g/R\omega^2 < 1$

$$\boxed{\omega^2 \geq g/R}$$







$\vartheta_0(\omega)$

Nekatere lastnosti L

- Lagrangeova funkcija je nedoločena do časovnega odvoda funkcije koordinat, če dodamo $\frac{d}{dt}F(q_1, q_2, \dots q_n, t)$ k L, bo ostala funkcija še vedno Lagrangeova f.

Torej,

poljubna ; $q_i(t)$

$$L' = L + \boxed{\frac{d}{dt}} F(q_1, q_2, \dots q_n, t)$$

$$\frac{d}{dt}\frac{\partial L'}{\partial \dot q_i} - \frac{\partial L'}{\partial q_i} = \frac{d}{dt}\frac{\partial L}{\partial \dot q_i} - \frac{\partial L}{\partial q_i} + \underbrace{\underbrace{\frac{d}{dt}\frac{\partial}{\partial \dot q_i}\underbrace{\left(\sum_j \frac{\partial F}{\partial q_j}\dot q_j + \frac{\partial F}{\partial t}\right)}_{=\frac{d}{dt}F}}_{=\frac{\partial F}{\partial q_i}} - \frac{\partial}{\partial q_i}\frac{d}{dt}F}_{=0}$$

$$\frac{d}{dt}\frac{\partial F}{\partial q_i} - \frac{\partial}{\partial q_i}\frac{d}{dt}F$$

$V \neq V(q_i)$

torej sledi končno

$$0 = \boxed{\frac{d}{dt}\frac{\partial L'}{\partial \dot q_i} - \frac{\partial L'}{\partial q_i} = \frac{d}{dt}\frac{\partial L}{\partial \dot q_i} - \cancel{\frac{\partial L}{\partial q_i}}} \overset{?}{=} 0$$

$L = 0$

$$\frac{d}{dt}\frac{\partial}{\partial \dot q_i}\left(\frac{d}{dt}F(q_1, q_2 \dots q_n, t)\right) - \frac{\partial}{\partial q_i}\left(\frac{d}{dt}F(q_1, q_2, \dots q_n, t)\right) = 0$$

•• mehanska podobnost

Če $L' = cL$ in $c \in \mathbb{R}$, je L' Lagrangeova funkcija, če je to tudi L. (očitno)

Zanimiv primer: potencial je homogena funkcija,

$$\rightarrow \; U(\alpha \vec r_1, \alpha \vec r_2, \dots \alpha \vec r_N) = \alpha^k U(\vec r_1, \vec r_2, \dots \vec r_N)$$

mi odmislimo od časa

$U = \frac{1}{r^2}$ $\quad k = -2$

velja torej npr.:

$\vec r_i \to \alpha \vec r_i = \vec r_i'$ (spreminjanje dolžinske enote)

$t \to \beta t$ (časovne)

$\vec v_i \to \frac{\alpha}{\beta} \vec v_i = \vec v_i'$

$T \to \left(\frac{\alpha}{\beta}\right)^2 T$

$U \to \alpha^k U$

če velja $\left(\frac{\alpha}{\beta}\right)^2 = \alpha^k$

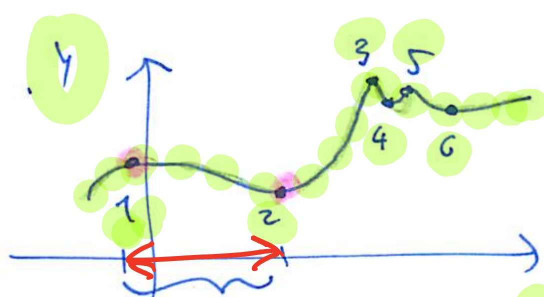
če se zgodi oz. $\beta = \alpha^{1-\frac{1}{2}k}$

$\frac{\alpha}{\beta}$ k

$\Rightarrow \; \Rightarrow \; L \to \underbrace{\alpha^k}_{c} L = L'$ trajektorije so torej geometrijsko podobne poti

$$\beta = \alpha^{1-\frac{1}{2}k}$$

npr.:



$x \to x'$, $t \to t'$

l

l'

l_{12} razdalja med točkama 1 in 2

delec to prepotuje v času t_{12}

skaliramo enote in bo veljalo:

$$\frac{t'}{t} = \left(\frac{l'}{l}\right)^{\boxed{1-\frac{1}{2}k}} ; \quad \frac{v'}{v} = \left(\frac{l'}{l}\right)^{\frac{1}{2}k} ; \quad \frac{E'}{E} = \left(\frac{l'}{l}\right)^{k}.$$

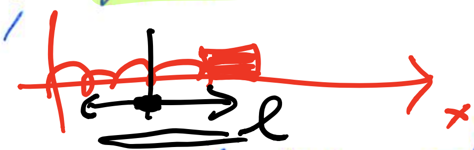
α

$$\beta = \frac{t'}{t} = \left(\frac{l'}{l}\right)^{1-\frac{1}{2}k}$$

$$\frac{E'}{E} = \alpha^k$$

primeri mehanske podobnosti: $k = 2^{31}$

1) $k = 2$ (harmonska sila); $V(x) \propto x^{2}$



$$\frac{t'}{t} = \left(\frac{l'}{l}\right)^{\overbrace{1-\frac{1}{2}k}^{0}} = 1$$

npr.: nihajni čas nihala neodvisen od amplitude

$\sqrt{\frac{2h}{g}}$

2) $k = 1$ (homogeno polje sil); $V(x) \propto x^{1}$

$1 - \frac{1}{2}k = \frac{1}{2}$ $\quad \frac{t'}{t} = \sqrt{l'/l}$ $\quad$ npr.: prosti pad - $t = \sqrt{\frac{2h}{g}}$

$V_g = -mgh^{1}$ $\quad k=1$

3) $k = -1$ (gravitacija, Coulomb); $V(x) \propto \frac{1}{x}$ $\quad k=-1$

$1 - \frac{1}{2}k = 3/2$ $\quad \frac{t'}{t} = \left(\frac{l'}{l}\right)^{3/2}$ $\quad$ 3. Keplerjev zakon $\boxed{t^2 \propto l^3}$ $\quad l \quad t$

4) virialni teorem (brez dokaza; naj je): gibanje delca naj bo omejeno in k

$$\Rightarrow \boxed{2\langle T\rangle = k\langle U\rangle} ; \quad \lim_{t_0\to\infty} \frac{1}{t_0}\int_0^{t_0} f(t)\,dt = \langle f\rangle$$

povprečje po času

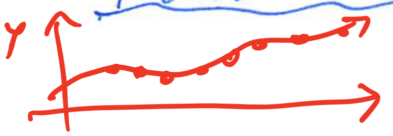
- nihalo ($k=2$): $\langle T\rangle = \langle U\rangle$
- $k=-1$: $2\langle T\rangle = -\langle U\rangle \geq 0$

$\Rightarrow \langle U\rangle < 0$ za omejena gibanja

$\langle U\rangle \leq 0 \Rightarrow$ elipse

(ne velja za npr. hiperbolične tire; delec prileti od daleč s pozitivno energijo - ni omejeno - in odleti v ∞)

Konstante gibanja (ohranjene količine)
(ponovno)



$\vec{r}(t) = (x(t), y(t))$

$\underline{q} = (q_1, q_2, \dots q_n)$

$F(\underline{q}, \dot{\underline{q}}, t)$ je konstanta gibanja, če

$$\boxed{\frac{dF}{dt} = \sum_i \frac{\partial F}{\partial q_i}\dot{q}_i + \sum_i \frac{\partial F}{\partial \dot{q}_i}\ddot{q}_i + \frac{\partial F}{\partial t} = 0} \quad !$$

vzdolž trajektorij ($\underline{q}$, ki zadoščajo Lagrangeovim enačbam).

• Naj Lagrangeeva funkcija ne odvisna od nekega q_j („ciklična" koordinata)
spregledljiva

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} = \frac{\partial L}{\partial q_j} = 0 \Rightarrow \boxed{p_j \overset{\text{def.}}{=} \frac{\partial L}{\partial \dot{q}_j} = \text{konst.}}$$

npr. gibalna količina, če
$V = \text{konst.}$
ali vrtilna količina,
če $V(\vec{r}) = V(|\vec{r}|)$ itd.

•• Naj L. funkcija ne odvisna od časa t:

$$\frac{\partial L}{\partial t} = 0 \Rightarrow H \overset{\text{def.}}{=} \sum_j \dot{q}_j \overbrace{\frac{\partial L}{\partial \dot{q}_j}}^{p_j} - L = \text{konst.}$$

$$= \sum_j p_j \dot{q}_j - L$$

Hamiltonova funkcija

dokaz:

$$\boxed{\frac{dH}{dt} = \sum_j \left(\ddot{q}_j \frac{\partial L}{\partial \dot{q}_j} + \dot{q}_j \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j}\dot{q}_j - \frac{\partial L}{\partial \dot{q}_j}\ddot{q}_j \right) - \frac{\partial L}{\partial t} =}$$

$=0$ (očitno)

$=0$ (Lagrange)

$$= -\frac{\partial L}{\partial t} = 0 \; ✓$$

$q_i(t)$

$$\frac{dH}{dt} = 0 \Rightarrow H = \text{konst}$$

Energija:

$$T = \sum_i \frac{1}{2} m_i \dot{\vec{r}}_i^{\,2}$$

$V \neq V(\dot{q}, t)$!

$$\boxed{V = V(q_1, q_2, \ldots q_n)}$$

$$\vec{r}_i = \vec{r}_i(q_1, q_2, \ldots q_n)$$

$i = 1, 2, \ldots N$

$j = 1, 2, \ldots n$

$$\dot{\vec{r}}_i = \sum_j \frac{\partial \vec{r}_i}{\partial q_j} \dot{q}_j \qquad \dot{\vec{r}}_i \cdot \dot{\vec{r}}_i$$

$$T = \sum_i \frac{1}{2} m_i \sum_{jk} \frac{\partial \vec{r}_i}{\partial q_j} \dot{q}_j \cdot \frac{\partial \vec{r}_i}{\partial q_k} \dot{q}_k =$$

$$= \sum_{jk} \boxed{\frac{1}{2} w_{jk}(q_1, q_2, \ldots q_n)}\, \dot{q}_j \dot{q}_k$$

~~najsplošnejša oblika~~ T

$j \overset{?}{=} k$

in $\quad w_{jk} = \sum_i m_i \frac{\partial \vec{r}_i}{\partial q_j} \cdot \frac{\partial \vec{r}_i}{\partial q_k} = w_{kj}$

$\dot{q}_j \dot{q}_k$

$\frac{\partial}{\partial \dot{q}_j}$

$$H = \sum_j \frac{\partial L}{\partial \dot{q}_j} \dot{q}_j - L =$$

$L = T - V$

$$= 2T \underbrace{- T + V}_{L} = \boxed{T + V = E} = \text{konst.}$$

energija

$$\sum_{kl} w_{kl} \dot{q}_k \dot{q}_l$$

: izrek Emmy Noether

Nöther

oe

Imamo enoparametrično poslikavo koordinat,

$s=0$

$$q_i(t) \to Q_i(s,t) \quad \text{in} \quad q_i(t) = Q_i(0,t); \quad s \in \mathbb{R}.$$

Ta poslikava ustreza zvezni simetriji L, ta (ni odvisno od s),

$$\frac{\partial L(Q_i(s,t), \dot{Q}_i(s,t), t)}{\partial s} = 0.$$

Naj to torej velja za L,

$$\boxed{0 = \frac{\partial}{\partial s} L} = \sum_i \frac{\partial L}{\partial Q_i}\frac{\partial Q_i}{\partial s} + \sum_i \frac{\partial L}{\partial \dot{Q}_i}\frac{\partial \dot{Q}_i}{\partial s} =$$

vzamemo: $s \to 0$

$$= \sum_i \frac{\partial L}{\partial q_i}\frac{\partial Q_i}{\partial s}\Big|_{s=0} + \sum_i \frac{\partial L}{\partial \dot{q}_i}\frac{\partial \dot{Q}_i}{\partial s}\Big|_{s=0} =$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0$$

Lagrange

$$= \sum_i \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}\right)\frac{\partial Q_i}{\partial s}\Big|_{s=0} + \sum_i \frac{\partial L}{\partial \dot{q}_i}\frac{\partial \dot{Q}_i}{\partial s}\Big|_{s=0} =$$

$$\frac{\partial \dot{Q}_i}{\partial s}\Big|_{s=0} = \frac{d}{dt}\frac{\partial Q_i}{\partial s}$$

$$= \frac{d}{dt}\sum_i \underbrace{\frac{\partial L}{\partial \dot{q}_i}}_{p_i}\frac{\partial Q_i}{\partial s}\Big|_{s=0} = 0$$

torej, $\frac{\partial L}{\partial \dot{q}_i} = p_i$

$$\boxed{\sum_i \underbrace{p_i}_{\vec{p}} \underbrace{\frac{\partial Q_i}{\partial s}\Big|_{s=0}}_{=\vec{n}} = \text{konst.}}$$

$q_i \to Q_i(s)$

$Q_i(s) \xrightarrow{s \to 0} q_i$

Vsaki zvezni simetriji Lagrangeove funkcije torej ustreza ena ohranjena količina

izrek Emmy Nöther

primeri (ponovno isto, malo drugače):

1) homogenost prostora

$$\vec r_i \to \vec Q_i(s,t) = \vec r_i + s\vec n \quad ; \quad s \in \mathbb{R}$$

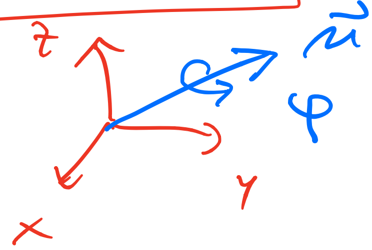
$$\frac{\partial \vec Q_i}{\partial s} = \vec n \qquad \vec n \text{ konstanten vektor}$$

naj velja $L(\vec r_i, \dot{\vec r}_i, t) = L(\vec Q_i, \dot{\vec Q}_i, t)$,

to pomeni, da je prostor homogen, se pravi, da translacija sistema ne spremeni enačb gibanja (≡ ni zunanjih sil),

$$\sum_i \frac{\partial L}{\partial \dot{\vec r}_i}\cdot \vec n = \Big(\sum_i \vec p_i\Big)\cdot \vec n \quad \text{za } \forall \vec n$$

$$\Rightarrow \boxed{\sum_i \vec p_i = \text{konst.}}$$



2) izotropnost prostora

$$\vec Q_i = \vec r_i + \underset{=s}{\varphi}\, \vec n \times \vec r_i$$

naj velja

$$L(\vec r_i, \dot{\vec r}_i; t) = L(\vec Q_i, \dot{\vec Q}_i, t),$$ se pravi,

da ostanejo enačbe gibanja enake v rotiranem sistemu,

$$\frac{\partial L}{\partial s} = 0 \Rightarrow \quad \sum_i \frac{\partial L}{\partial \dot{\vec r}_i}\cdot \underbrace{(\vec n \times \vec r)}_{\frac{\partial \vec Q_i}{\partial \varphi}} = \sum_i \vec n\cdot(\vec r_i \times \vec p_i) = \vec n \cdot \vec L$$

$\vec r_i \cdot \frac{\partial \vec Q_i}{\partial \varphi}$

za $\forall \vec n \Rightarrow \vec L = \text{konst.}$

vrtilna količina

3) homogenost časa: L ni odvisna od časa, $\frac{\partial L}{\partial t} = 0$.

$$t \to t + t_0 \Rightarrow \boxed{H = \text{konst.}} = E$$

$$t_0 = s \in \mathbb{R}$$