

Kardinalna Aritmetika: Izpit

June 9, 2020

This exam contains four questions. You are required to answer all four. All answers must be justified. You have 120 minutes to complete the exam. You may answer in English or in Slovene. Good luck!

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Seat (3.05)

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Name and surname

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Question 1 (25 marks)

A set S is said to be *Tarski finite* if every nonempty subset $X \subseteq \mathcal{P}(A)$ has a *maximal element*. (That is, there exists $A \in X$ such that there is no $B \in X$ with $A \subset B$.)

a) (5 marks) Prove that $\mathbb{N}$ is not Tarski finite.

b) (5 marks) Prove that if a set A is infinite then it is not Tarski finite.

c) (15 marks) Prove that a set is finite if and only if it is Tarski finite.

Question 2 (25 marks)

a) (8 marks) Prove that an ordinal α is a limit ordinal if and only if, for every $\beta < \alpha$, it holds that $\beta + 1 < \alpha$.

b) (10 marks) Prove that every ordinal α is either a limit ordinal or a successor ordinal but not both.

(More space for your answer to Question 2b)

c) (7 marks) Prove that, for all ordinals α, β , if β is a limit ordinal then so is $\alpha \cdot \beta$.

Question 3 (25 marks)

a) (10 marks) Prove the cardinal equality

$$(|X|^{|Y|})^{|Z|} = |X|^{|Y| \cdot |Z|} .$$

b) (10 marks) For each of the two statements below state whether or not it is provable in set theory with AC. In the case of a claim of provability, give an outline of the proof, and explain where AC is used. In the case of a claim of unprovability, justify your answer by reference to set-theoretic hypotheses that are known to be unprovable.

(i) $\aleph_{\alpha+1} \leq 2^{\aleph_\alpha}$

(ii) $2^{\aleph_\alpha} \leq \aleph_{\alpha+1}$

c) (5 marks) Prove the cardinal equality $\aleph_{\alpha+1}^{\aleph_\alpha} = 2^{\aleph_\alpha}$ in set theory with AC.

Question 4 (25 marks)

In the space below and on the opposite page, answer **exactly one** of the two questions below.

1. Consider the property below.

For every surjective function $e : A \rightarrow B$ there exists a function $s : B \rightarrow A$ such that the composite function $e \circ s : B \rightarrow B$ is the identity function.

Prove the following three statements.

- (a) The above property holds for all sets A, B with A countable.
- (b) The above property holds for all sets A, B with B countable.
- (c) The above property holds for all sets A, B .

In each case, if you make use of a choice axiom, you are required to also prove that the choice axiom used is itself equivalent to the statement being proved.

2. A function $f : \downarrow\alpha \rightarrow \downarrow\beta$ is said to be *cofinal* if, for every $\beta' < \beta$, there exists $\alpha' < \alpha$ such that $f(\alpha') \geq \beta'$.

- (a) Prove that if $f : \downarrow\alpha \rightarrow \downarrow\beta$ is a strictly increasing cofinal function then $\text{cf}(\beta) \leq \text{cf}(\alpha)$, where $\text{cf}(\alpha)$ is the *cofinality* of α .
- (b) Prove that if $f : \downarrow\alpha \rightarrow \downarrow\beta$ is a cofinal function then there exists a strictly increasing cofinal $f' : \downarrow\alpha' \rightarrow \downarrow\beta$ such that $\text{image}(f') \subseteq \text{image}(f)$, for some ordinal $\alpha' \leq \alpha$.
- (c) Prove that if $f : \downarrow\alpha \rightarrow \downarrow\beta$ is a strictly increasing cofinal function then $\text{cf}(\alpha) \leq \text{cf}(\beta)$. (Hint: start by defining a cofinal function $g : \downarrow\beta \rightarrow \downarrow\alpha$ that satisfies $f(g(\beta')) \geq \beta'$ for every $\beta' < \beta$.)

(Write your answer here.)

(More space for your answer to Question 4.)