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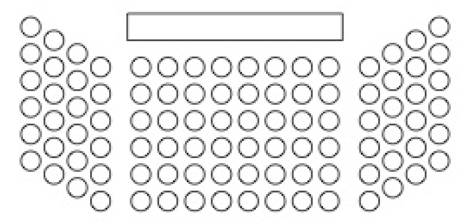
2. kolokvij iz Algebre 1 za finančnike

2. 3. 2022

Veliko uspeha!



Sedež (2.01)

Ime in priimek ______________________ Vpisna številka

1. naloga

Naj bo

$$U_a = \mathrm{Lin}\{[2,-1,a,1,7]^T, [1,-1,a+1,3,7]^T, [a,1,0,-1,1]^T, [4,-1,a-2,-3,7]^T\},$$
$$V = \{[x,y,z,u,v]^T \in \mathbb{R}^5; x+2y-z+u=0, 2x-y+2z-v=0\}.$$

a) V odvisnosti od $a \in \mathbb{R}$ določi bazo za U_a.

b) Določi kake baze za V, $U_1 \cap V$ in $U_1 + V$.

c) Določi vse $a \in \mathbb{R}$, da je $U_a \cap V = \{0\}$, in vse $a \in \mathbb{R}$, da je $\dim(U_a \cap V) = 1$.

a)
$$\begin{bmatrix} 1 & -1 & a+1 & 3 & 7 \\ 2 & -1 & a & 1 & 7 \\ a & 1 & 0 & -1 & 1 \\ 4 & -1 & a-2 & -3 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & a+1 & 3 & 7 \\ 1 & 0 & -1 & -2 & 0 \\ a+1 & 0 & a+1 & 2 & 8 \\ 3 & 0 & -3 & -6 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & -1 & a+2 & 5 & 7 \\ 1 & 0 & -1 & -2 & 0 \\ 0 & 0 & 2a+2 & 2a+4 & 8 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Za vsak $a \in \mathbb{R}$ je rang matrike enak 3, zato je baza za U_a na primer $\{[1,0,-1,-2,0]^T, [0,-1,a+2,5,7]^T, [0,0,2a+2,2a+4,8]^T\}$.

b) $\mathcal{B}_V$: Za parametre vzamemo x,y,z in dobimo $\mathcal{B}_V = \{[1,0,0,-1,2]^T, [0,1,0,-2,-1]^T, [0,0,1,1,2]^T\}$.

Iz a) vemo, da je $\{[1,0,-1,-2,0]^T, [0,-1,3,5,7]^T, [0,0,4,6,8]^T\}$ baza za U_1, ki jo "polepšamo":

$$\begin{bmatrix} 1 & 0 & -1 & -2 & 0 \\ 0 & -1 & 3 & 5 & 7 \\ 0 & 0 & 4 & 6 & 8 \end{bmatrix} \begin{matrix} \\ \cdot(-1) \\ :4 \end{matrix} \sim \begin{bmatrix} 1 & 0 & -1 & -2 & 0 \\ 0 & 1 & -3 & -5 & -7 \\ 0 & 0 & 1 & 3/2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -1/2 & 2 \\ 0 & 1 & 0 & -1/2 & -1 \\ 0 & 0 & 1 & 3/2 & 2 \end{bmatrix}$$

$\mathcal{B}_{U_1} = \{[1,0,0,-1/2,2]^T, [0,1,0,-1/2,-1]^T, [0,0,1,3/2,2]^T\}$.

$\mathcal{B}_{U_1+V}$:
$$\begin{bmatrix} 1 & 0 & 0 & -1 & 2 \\ 0 & 1 & 0 & -2 & -1 \\ 0 & 0 & 1 & 1 & 2 \\ 1 & 0 & 0 & -1/2 & 2 \\ 0 & 1 & 0 & -1/2 & -1 \\ 0 & 0 & 1 & 3/2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -1 & 2 \\ 0 & 1 & 0 & -2 & -1 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1/2 & 0 \\ \cancel{0} & \cancel{0} & \cancel{0} & \cancel{3/2} & \cancel{0} \\ \cancel{0} & \cancel{0} & \cancel{0} & \cancel{1/2} & \cancel{0} \end{bmatrix} \begin{matrix} \\ \\ \\ \cdot 2 \\ \\ \\ \end{matrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$\Rightarrow \mathcal{B}_{U_1+V} = \{[1,0,0,0,2]^T, [0,1,0,0,-1]^T, [0,0,1,0,2]^T, [0,0,0,1,0]^T\}$.

$\mathcal{B}_{U_1 \cap V}$: $\dim(U_1 \cap V) = \dim U_1 + \dim V - \dim(U_1+V) = 3+3-4 = 2$

$$\alpha_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \\ 2 \end{bmatrix} + \beta_1 \begin{bmatrix} 0 \\ 1 \\ 0 \\ -2 \\ -1 \end{bmatrix} + \gamma_1 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 2 \end{bmatrix} = \alpha_2 \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1/2 \\ 2 \end{bmatrix} + \beta_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1/2 \\ -1 \end{bmatrix} + \gamma_2 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 3/2 \\ 2 \end{bmatrix}$$

$\Rightarrow \alpha_2 = \alpha_1, \beta_2 = \beta_1, \gamma_2 = \gamma_1$

$\Rightarrow -\alpha_1 - 2\beta_1 + \gamma_1 = -1/2\,\alpha_1 - 1/2\,\beta_1 + 3/2\,\gamma_1 \Rightarrow 0 = \alpha_1 + 3\beta_1 + \gamma_1$

$\Rightarrow 2\alpha_1 - \beta_1 + 2\gamma_1 = 2\alpha_1 - \beta_1 + 2\gamma_2$ ✓

$\alpha_1 = 1, \beta_1 = 0 \Rightarrow \gamma_1 = -1 \Rightarrow [1,0,0,-1,2]^T - [0,0,1,1,2]^T = [1,0,-1,-2,0]^T$

$\alpha_1 = 0, \beta_1 = 1 \Rightarrow \gamma_1 = -3 \Rightarrow [0,1,0,-2,-1]^T - 3[0,0,1,1,2]^T = [0,1,-3,-5,-7]^T \Rightarrow \mathcal{B}_{U_1 \cap V} = \{[1,0,-1,-2,0]^T, [0,1,-3,-5,-7]^T\}$

c) Ker je $\dim U_a = 3 = \dim V$, je $\dim(U_a \cap V) = \dim U_a + \dim V - \dim(U_a+V) \geq 3+3-5 = 1 \Rightarrow U_a \cap V = \{0\}$ za noben a

$\dim(U_a \cap V) = 1 \Leftrightarrow \dim(U_a+V) = 5$:

$$\dim(U_a+V) = \mathrm{rang}\begin{bmatrix} 1 & 0 & 0 & -1 & 2 \\ 0 & 1 & 0 & -2 & -1 \\ 0 & 0 & 1 & 1 & 2 \\ 1 & 0 & -1 & -2 & 0 \\ 0 & -1 & a+2 & 5 & 7 \\ 0 & 0 & 2a+2 & 2a+4 & 8 \end{bmatrix} = \mathrm{rang}\begin{bmatrix} 1 & 0 & 0 & -1 & 2 \\ 0 & 1 & 0 & -2 & -1 \\ 0 & 0 & 1 & 1 & 2 \\ \cancel{0} & \cancel{0} & \cancel{-1} & \cancel{-1} & \cancel{-2} \\ 0 & 0 & a+2 & 3 & 6 \\ 0 & 0 & 2a+2 & 2a+4 & 8 \end{bmatrix} = \mathrm{rang}\begin{bmatrix} 1 & 0 & 0 & -1 & 2 \\ 0 & 1 & 0 & -2 & -1 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & a-1 & 0 & 0 \\ 0 & 0 & -2 & 0 & -4a \end{bmatrix} = 3 + \mathrm{rang}\begin{bmatrix} a-1 & 0 \\ -2 & -4a \end{bmatrix} = \begin{cases} 4, & a = 0,1, \\ 5, & \text{sicer.} \end{cases}$$

Odg.: $\dim(U_a \cap V) = 1$ natanko tedaj, ko $a \in \mathbb{R} \setminus \{0,1\}$.

2. naloga

Naj bo $q_i(x) = x^2 + (x+1)^i$, $\mathcal{B} = \{q_0, q_1, q_2\}$ in $p(x) = x$. Naj bo $A_t\colon \mathbb{R}_2[x] \to \mathbb{R}_2[x]$ linearna preslikava podana s predpisom

$$A_t(q_i)(x) = tq_i(x) + q_i(-1)q_i'(x)$$

za $i = 0, 1, 2$.

a) Izračunaj $A_t(p)$.

b) Določi vse $t \in \mathbb{R}$, da je A_t injektivna.

c) Zapiši matriko za A_2, ki pripada linearni preslikavi v bazi $\mathcal{B}$, ter določi kaki bazi za $\ker A_2$ in $\operatorname{im} A_2$.

$q_0(x) = x^2+1$, $q_1(x) = x^2+x+1$, $q_2(x) = 2x^2+2x+1$

$A_t(q_0)(x) = t(x^2+1) + 2(2x) = tx^2+4x+t$

$A_t(q_1)(x) = t(x^2+x+1) + 1(2x+1) = tx^2 + (t+2)x + t+1$

$A_t(q_2)(x) = t(2x^2+2x+1) + 1(4x+2) = 2tx^2 + (2t+4)x + t+2$

(Opomba: Predpis $A_t(q)(x) = tq(x) + q(-1)q'(x), \forall q \in \mathbb{R}_2[x]$, $\underline{\underline{\text{ni}}}$ linearna preslikava, zato ne moremo vrednost preslikave v polinomu p izračunati kot $A_t(p)(x) = tp(x) + p(-1)p'(x)$.)

a) $p = q_1 - q_0 \Rightarrow A_t(p)(x) = A_t(q_1)(x) - A_t(q_0)(t) = \underline{\underline{(t-2)x+1}}$

b) A_t je injektivna natanko takrat, ko je surjektivna.

Ogrodje za $\operatorname{im} A_t$ je $\{A_t(q_0), A_t(q_1), A_t(q_2)\}$:

$$\begin{bmatrix} t & 4 & t \\ t & t+2 & t+1 \\ 2t & 2t+4 & t+2 \end{bmatrix} \sim \begin{bmatrix} t & 4 & t \\ 0 & t-2 & 1 \\ 0 & 2t-4 & 2-t \end{bmatrix} \sim \begin{bmatrix} t & 4 & t \\ 0 & t-2 & 1 \\ 0 & 0 & -t \end{bmatrix} =: X$$

A_t surjektivna $\Leftrightarrow \operatorname{im} A_t = \mathbb{R}_2[x] \Leftrightarrow \operatorname{rang} X = 3 \Leftrightarrow t \neq 0, 2$

Odg: $\underline{\underline{A_t \text{ je injektivna} \Leftrightarrow t \in \mathbb{R} \setminus \{0,2\}}}$

Nalogo lahko rešujemo tudi tako, da izračunamo $\ker A_t$, saj je A_t injektivna $\Leftrightarrow \ker A_t = 0$:

Naj bo $q = \alpha q_0 + \beta q_1 + \gamma q_2 \in \ker A_t \Rightarrow 0 = A_t(q)(x) = \alpha(tx^2+4x+t) + \beta(tx^2+(t+2)x+t+1) + \gamma(2tx^2+(2t+4)x+t+2)$.

Pogledamo koeficiente pri potencah spremenljivke x:

x^2: $\alpha t + \beta t + 2\gamma t = 0$

x: $4\alpha + \beta(t+2) + \gamma(2t+4) = 0$

1: $t\alpha + \beta(t+1) + \gamma(t+2) = 0$

Jedro $\ker A_t$ je trivialno $\Leftrightarrow$ sistem ima natanko eno rešitev $\Leftrightarrow$ determinanta homogenega sistema je neničelna:

$$0 \neq \det \begin{bmatrix} t & t & 2t \\ 4 & t+2 & 2t+4 \\ t & t+1 & t+2 \end{bmatrix} = t\begin{vmatrix} 1 & 1 & 2 \\ 4 & t+2 & 2t+4 \\ 0 & 1 & 2-t \end{vmatrix} = t\begin{vmatrix} 1 & 1 & 2 \\ 0 & t-2 & 2t-8 \\ 0 & 1 & 2-t \end{vmatrix} = t(t-2)\begin{vmatrix} 1 & 2 \\ 1 & 2-t \end{vmatrix} = t(t-2)\begin{vmatrix} 1 & 2 \\ 0 & -t \end{vmatrix} = -t^2(t-2) \Rightarrow \underline{\underline{t \neq 0,2}}$$

c) Naj bo $\mathcal{S} = \{x^2, x, 1\}$

$$(A_2)_{\mathcal{SB}} = \begin{array}{c} \\ \\ \\ \\ \end{array}\begin{array}{cl} \begin{bmatrix} 2 & 2 & 4 \\ 4 & 4 & 8 \\ 2 & 3 & 4 \end{bmatrix} & \begin{matrix} \leftarrow x^2 \\ \leftarrow x \\ \leftarrow 1 \end{matrix} \\ \begin{matrix} \uparrow & \uparrow & \uparrow \\ A(q_0) & A(q_1) & A(q_2) \end{matrix} & \end{array}$$

$$P_{\mathcal{SB}} = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

$$P_{\mathcal{BS}} = P_{\mathcal{SB}}^{-1}: \left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & -1 & -1 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & -2 & 1 & 2 \\ 0 & 0 & 1 & 1 & 0 & -1 \end{array}\right]$$

$$(A_2)_{\mathcal{BB}} = P_{\mathcal{BS}}(A_2)_{\mathcal{SB}} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 1 & 2 \\ 1 & 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 2 & 4 \\ 4 & 4 & 8 \\ 2 & 3 & 4 \end{bmatrix} = \underline{\underline{\begin{bmatrix} -2 & -2 & -4 \\ 4 & 6 & 8 \\ 0 & -1 & 0 \end{bmatrix}}}$$

$$\mathcal{B}_{\operatorname{im} A_2}: \begin{bmatrix} 2 & 4 & 2 \\ 2 & 4 & 3 \\ 4 & 8 & 4 \end{bmatrix} \sim \begin{bmatrix} 2 & 4 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \underline{\underline{\mathcal{B}_{\operatorname{im} A_2} = \{x^2+2x, 1\}}}$$

$\Rightarrow \dim \ker A_2 = 3 - \dim \operatorname{im} A_2 = 1$. Ker je $A_2(q_1) - 2A_2(q_0) = 0$, je $\mathcal{B}_{\ker A_2} = \{q_1 - 2q_0\} = \{-x^2+x-1\}$

3. naloga

Interval $G := (-\frac{\pi}{2}, \frac{\pi}{2})$ opremimo z operacijo $\circ$, definirano s predpisom

$$x \circ y := \operatorname{arctg}(\operatorname{tg} x + \operatorname{tg} y).$$

a) Dokaži, da je $(G, \circ)$ Abelova grupa, pri čemer ne pozabi preveriti, da je operacija $\circ$ notranja.

b) Dokaži, da sta grupi $(G, \circ)$ in $(\mathbb{R}, +)$ izomorfni.

c) V grupi G reši enačbo $x \circ x \circ x = \frac{\pi}{3}$.

a) $\circ$ je notranja operacija

$x, y \in (-\frac{\pi}{2}, \frac{\pi}{2}) \Rightarrow \operatorname{tg} x, \operatorname{tg} y \in \mathbb{R} \Rightarrow \operatorname{tg} x + \operatorname{tg} y \in \mathbb{R} \Rightarrow \operatorname{arctg}(\operatorname{tg} x + \operatorname{tg} y) \in (-\frac{\pi}{2}, \frac{\pi}{2})$.

$(G, \circ)$ je Abelova grupa

$(x \circ y) \circ z = \operatorname{arctg}(\operatorname{tg}(x \circ y) + \operatorname{tg} z) = \operatorname{arctg}(\operatorname{tg}(\operatorname{arctg}(\operatorname{tg} x + \operatorname{tg} y)) + \operatorname{tg} z)$

$= \operatorname{arctg}((\operatorname{tg} x + \operatorname{tg} y) + \operatorname{tg} z) = \operatorname{arctg}(\operatorname{tg} x + \operatorname{tg} y + \operatorname{tg} z)$

Na podoben način izračunamo $x \circ (y \circ z) = \operatorname{arctg}(\operatorname{tg} x + (\operatorname{tg} y + \operatorname{tg} z))$

$x \circ y = y \circ x$ $\qquad x \circ y = \operatorname{arctg}(\operatorname{tg} x + \operatorname{tg} y) = \operatorname{arctg}(\operatorname{tg} y + \operatorname{tg} x) = y \circ x.$

enota za $\circ$? $\qquad x \circ e = x = e \circ x$ (ker G komut., je $e \circ x = x \circ e$)

$x = x \circ e = \operatorname{arctg}(\operatorname{tg} x + \operatorname{tg} e) = x \iff \operatorname{tg} x + \operatorname{tg} e = \operatorname{tg} x$

$\iff \operatorname{tg} e = 0 \iff x = k\pi;\ k \in \mathbb{Z} \iff x = 0 \in G.$

Inverz za x. $\qquad x \circ y = 0 = y \circ x \iff x \circ y = 0 \iff \operatorname{arctg}(\operatorname{tg} x + \operatorname{tg} y) = 0$

$\iff \operatorname{tg} x + \operatorname{tg} y = 0 \iff \operatorname{tg} y = -\operatorname{tg} x = \operatorname{tg}(-x)$

$\iff y = -x + k\pi,\ k \in \mathbb{Z} \iff y = -x$, da $y \in G$.

b) $f(x) = \operatorname{tg} x$; $f : (-\pi/2, \pi/2) \to \mathbb{R}$ bijektivna.

$f(x \circ y) = \operatorname{tg}(\operatorname{arctg}(\operatorname{tg} x + \operatorname{tg} y)) = \operatorname{tg} x + \operatorname{tg} y = f(x) + f(y)$.

c) $x \circ x \circ x = (x \circ x) \circ x = \operatorname{arctg}(\operatorname{tg}(x \circ x) + \operatorname{tg} x) = \operatorname{arctg}(\operatorname{tg}(\operatorname{arctg}(\operatorname{tg} x + \operatorname{tg} x)) + \operatorname{tg} x)$

$= \operatorname{arctg}(3 \operatorname{tg} x)$

Enačba je torej $\operatorname{arctg}(3 \operatorname{tg} x) = \frac{\pi}{3}$

$\iff 3 \operatorname{tg} x = \operatorname{tg} \frac{\pi}{3} = \sqrt{3} \iff \operatorname{tg} x = \frac{\sqrt{3}}{3} \iff x = \frac{\pi}{6} + k\pi;\ k \in \mathbb{Z}.$

ker $x \in G$, je $\boxed{x = \frac{\pi}{6}}$.

4. naloga

V odvisnosti od naravnega števila n izračunaj determinanto

$$D_n = \begin{vmatrix} 4 & \sqrt{3} & 0 & 0 & \cdots & 0 & 0 & 0 \\ \sqrt{3} & 4 & \sqrt{3} & 0 & \cdots & 0 & 0 & 0 \\ 0 & \sqrt{3} & 4 & \sqrt{3} & \cdots & 0 & 0 & 0 \\ 0 & 0 & \sqrt{3} & 4 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 4 & \sqrt{3} & 0 \\ 0 & 0 & 0 & 0 & \cdots & \sqrt{3} & 4 & \sqrt{3} \\ 0 & 0 & 0 & 0 & \cdots & 0 & \sqrt{3} & 4 \end{vmatrix}$$

velikosti $n \times n$.

Razvoj po 1. vrstici.

$$D_n = 4 \cdot \underbrace{\begin{vmatrix} 4 & \sqrt{3} & & \\ \sqrt{3} & 4 & \sqrt{3} & \\ & \sqrt{3} & 4 & \ddots \\ & & \ddots & \sqrt{3} \\ & & \sqrt{3} & 4 \end{vmatrix}}_{n-1} \Bigg\} n-1 \; - \sqrt{3} \cdot \underbrace{\begin{vmatrix} \sqrt{3} & \sqrt{3} & & \\ 0 & 4 & \sqrt{3} & \\ 0 & \sqrt{3} & 4 & \ddots \\ & & \ddots & \sqrt{3} \\ & & & 4 \\ & & & \sqrt{3} \end{vmatrix}}_{n-1} \Bigg\} n-1$$

$\Rightarrow D_{n-1}$

Razvijemo še drugo determinanto po 1. stolpcu:

$$D_n = 4D_{n-1} - \sqrt{3} \cdot \sqrt{3} \cdot D_{n-2} = 4D_{n-1} - 3D_{n-2}$$

$$\Rightarrow \lambda^2 - 4\lambda + 3 = 0 \qquad \lambda_1 = 3, \ \lambda_2 = 1$$

$$\Rightarrow D_n = A \cdot 3^n + B \cdot 1^n \ ; \quad D_1 = 4 \text{ in } D_2 = 13$$

$$4 = 3A + B$$
$$13 = 9A + B$$
$$\Rightarrow 6A = 9 \Rightarrow A = \frac{3}{2}$$
$$\Rightarrow B = 4 - 3A = 4 - \frac{9}{2} = -\frac{1}{2}$$

$$D_n = \frac{1}{2}\left(3^{n+1} - 1\right)$$