LINEARNA ALGEBRA 2020/21
1. VAJE: 5. 10. ~~2019~~ 2020

1. Naj bodo $A = (0, 2, 3), B = (4, -2, 1)$ in $C = (8, 1, -3)$.

   1.1. Izračunaj vektorje $\vec{a} = \overrightarrow{AB}, \vec{b} = \overrightarrow{BC}$ in $\vec{c} = \overrightarrow{CA}$.

   1.2. Izračunaj vektorja $\frac{1}{2}\vec{a} - 2\vec{b}$ in $\vec{a} + \vec{b} + \vec{c}$.

   1.3. Ali sta vektorja $(-6, -8, 7)$ in $(3, -1, 0)$ linearni kombinaciji vektorjev $\vec{a}$ in $\vec{b}$?

2. V pravilnem šestkotniku $ABCDEF$ označimo $\vec{a} = \overrightarrow{AB}$ in $\vec{b} = \overrightarrow{BC}$. Z $\vec{a}$ in $\vec{b}$ izrazi: $\overrightarrow{EF}, \overrightarrow{CD}, \overrightarrow{AE}, \overrightarrow{BE}, \overrightarrow{CF}$ in $\overrightarrow{BF}$.

3. V trapezu $ABCD$ velja $\overrightarrow{AB} = 2\overrightarrow{DC}$. Naj bo E razpoločišče roba AD in F presečišče daljic AC in EB. Z $\vec{a} = \overrightarrow{AB}$ in $\vec{b} = \overrightarrow{AD}$ izrazi $\overrightarrow{AF}$.

4. V paralelogramu $ABCD$ označimo $\vec{a} = \overrightarrow{AB}$ in $\vec{b} = \overrightarrow{AD}$. Naj bosta E in F razpolovišči daljic AB in BC ter G in H presečišči daljic DE in DF z AC. Z $\vec{a}$ in $\vec{b}$ izrazi $\overrightarrow{AG}, \overrightarrow{GH}$ in $\overrightarrow{HC}$.

5. V kocki $ABCDA'B'C'D'$ izrazi presečišče trikotnika $AB'D'$ in diagonale $A'C$.

6. Naj bodo A_1, A_2 in A_3 poljubne točke ter a_1, a_2 in a_3 realna števila, za katera velja $a_1 +_2 a + a_3 = 1$. Pokaži, da obstaja natanko ena točka A, za katero velja $a_1\overrightarrow{AA_1} + a_2\overrightarrow{AA_2} + a_3\overrightarrow{AA_3} = 0$ in $\overrightarrow{BA} = a_1\overrightarrow{BA_1} + a_2\overrightarrow{BA_2} + a_3\overrightarrow{BA_3}$ za katerokoli točko B.

7. Dokaži, da se težiščnice trikotnika sekajo v eni točki.

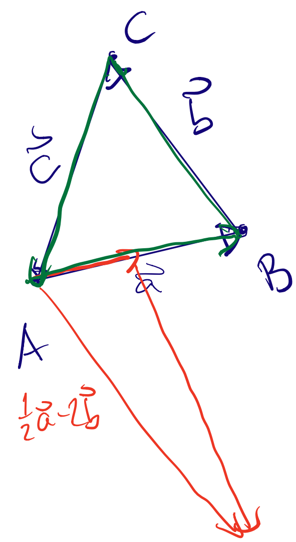
a) Konstruiraj težišče n točk v ravnini, če imaš podano težišče prvih $n-1$ točk in n-to točko.

1. Naj bodo $A = (0, 2, 3)$, $B = (4, -2, 1)$ in $C = (8, 1, -3)$.

   1.1. Izračunaj vektorje $\vec{a} = \overrightarrow{AB}, \vec{b} = \overrightarrow{BC}$ in $\vec{c} = \overrightarrow{CA}$.

   1.2. Izračunaj vektorja $\frac{1}{2}\vec{a} - 2\vec{b}$ in $\vec{a} + \vec{b} + \vec{c}$.

   1.3. Ali sta vektorja $(-6, -8, 7)$ in $(3, -1, 0)$ linearni kombinaciji vektorjev $\vec{a}$ in $\vec{b}$?



$\vec{a} = \overrightarrow{AB} = (4,-2,1) - (0,2,3) = (4,-4,-2)$

$\vec{b} = (8,1,-3) - (4,-2,1) = (4,3,-4)$

$\vec{c} = (0,2,3) - (8,1,-3) = (-8,1,6)$

$\frac{1}{2}(4,-4,-2) = (2,-2,-1)$

$2(4,3,-4) = (8,6,-8)$

$\Rightarrow \boxed{\frac{1}{2}\vec{a} - 2\vec{b} = (-6,-8,7)}$

$\vec{a} + \vec{b} + \vec{c} = (4+4-8, -4+3+1, -2-4+6) = (0,0,0)$

$\vec{x}$ je lin. kombinacija $\{y_i \mid i=1..n\}$ $\quad x = \sum_{i=1}^{n} \alpha_i y_i = \alpha_1 y_1 + \alpha_2 y_2 + \dots + \alpha_n y_n$

1.3. Ali sta vektorja $(-6, -8, 7)$ in $(3, -1, 0)$ linearni kombinaciji vektorjev $\vec{a}$ in $\vec{b}$?

$\alpha(4,-4,-2) + \beta(4,3,-4) = (-6,-8,7)$

$4\alpha + 4\beta = -6$
$-4\alpha + 3\beta = -8$
$-2\alpha - 4\beta = 7$ ✓

$\Rightarrow 7\beta = -14 \rightsquigarrow \beta = -2$

$4\alpha - 8 = -6$

$4\alpha = 2 \quad \alpha = \frac{1}{2}$

$\frac{1}{2}\vec{a} - 2\vec{b} = (-6,-8,7)$

$-1 + 8 = 7$

$\alpha'(4,-4,-2) + \beta'(4,3,-4) = (3,-1,0)$ $\rightsquigarrow$ $(3,-1,0)$ ni lin. kombinacija $\vec{a}, \vec{b}$

• $4\alpha' + 4\beta' = 3$

$-4\alpha' + 3\beta' = -1$

• $-2\alpha' - 4\beta' = 0$

Sistem nima rešitve

$2\alpha' = -4\beta'$

$\alpha' = -2\beta'$

$-8\beta' + 4\beta' = 3$

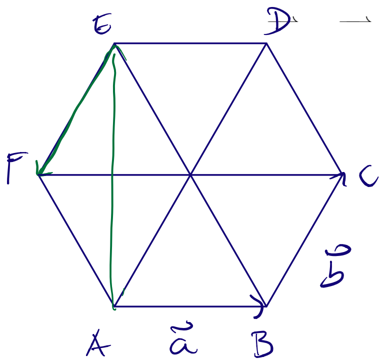
$$\boxed{\begin{aligned} \beta' &= -\frac{3}{4} \\ \alpha' &= \frac{6}{4} \end{aligned}}$$

$-\frac{24}{4} - \frac{9}{4} \stackrel{?}{=} -1$

$-\frac{33}{4} \neq -1$

$\vec{a}, \vec{b}, (3,-10)$ so lin. neodvisni $\Rightarrow$ so baza.

2. V pravilnem šestkotniku $ABCDEF$ označimo $\vec{a} = \overrightarrow{AB}$ in $\vec{b} = \overrightarrow{BC}$. Z $\vec{a}$ in $\vec{b}$ izrazi: $\overrightarrow{EF}, \overrightarrow{CD}, \overrightarrow{AE}, \overrightarrow{BE}, \overrightarrow{CF}$ in $\overrightarrow{BF}$.



$\overrightarrow{EF} = -\vec{b}$

$\overrightarrow{CD} = \vec{b} - \vec{a}$

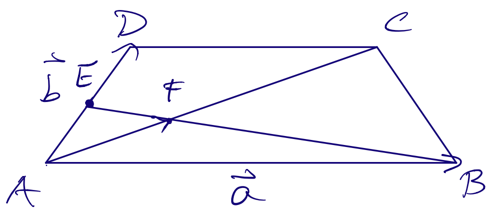
$\overrightarrow{AE} = 2\vec{b} - \vec{a}$

$\overrightarrow{BE} = 2\vec{b} - 2\vec{a}$

$\overrightarrow{CF} = -2\vec{a}$

$\overrightarrow{BF} = \vec{b} - 2\vec{a}$

3. V trapezu $ABCD$ velja $\overrightarrow{AB} = 2\overrightarrow{DC}$. Naj bo E razpoločišče roba AD in F presečišče daljic AC in EB. Z $\vec{a} = \overrightarrow{AB}$ in $\vec{b} = \overrightarrow{AD}$ izrazi $\overrightarrow{AF}$.



$$\overrightarrow{DC} = \frac{1}{2}\vec{a}$$

$$\overrightarrow{AF} = \alpha\vec{a} + \beta\vec{b}$$

$$\overrightarrow{AF} \parallel \overrightarrow{AC} \qquad \overrightarrow{AC} = \frac{1}{2}\vec{a} + \vec{b} \qquad \overrightarrow{EB} = \left(-\frac{1}{2}\vec{b} + \vec{a}\right)$$

$$\underline{\overrightarrow{AF}} = x\overrightarrow{AC} = \underline{x\left(\frac{1}{2}\vec{a} + \vec{b}\right)}$$

$$\underline{\overrightarrow{AF}} = \overrightarrow{AE} + y\overrightarrow{EB} = \frac{1}{2}\vec{b} + y\left(\vec{a} - \frac{1}{2}\vec{b}\right)$$

$$x\frac{1}{2}\vec{a} + x\vec{b} = y\vec{a} + \left(\frac{1}{2} - \frac{1}{2}y\right)\vec{b}$$

$$\underline{\left(\frac{1}{2}x - y\right)}\vec{a} + \underline{\left(x + \frac{1}{2}y - \frac{1}{2}\right)}\vec{b} = 0$$

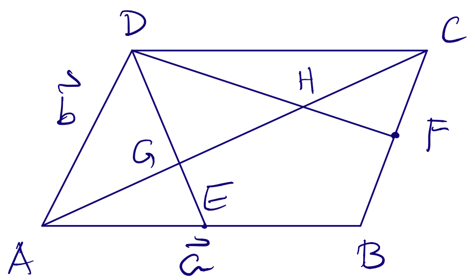
$$\frac{1}{2}x = y$$

$$x + \frac{1}{2}y = \frac{1}{2}$$

$$x + \frac{1}{4}x = \frac{1}{2} \qquad \frac{5}{4}x = \frac{1}{2} \Rightarrow x = \frac{2}{5}$$

$$\overrightarrow{AF} = \frac{2}{5}\left(\frac{1}{2}\vec{a} + \vec{b}\right) = \underline{\frac{1}{5}\vec{a} + \frac{2}{5}\vec{b}}$$

4. V paralelogramu $ABCD$ označimo $\vec{a} = \overrightarrow{AB}$ in $\vec{b} = \overrightarrow{AD}$. Naj bosta E in F razpolovišči daljic AB in BC ter G in H presečišči daljic DE in DF z AC. Z $\vec{a}$ in $\vec{b}$ izrazi $\overrightarrow{AG}, \overrightarrow{GH}$ in $\overrightarrow{HC}$.



$\overrightarrow{AG}$, $\overrightarrow{GH}$, $\overrightarrow{HC}$

$$\overrightarrow{AG} = \alpha \overrightarrow{AC} = \alpha(\vec{a} + \vec{b})$$
$$\overrightarrow{AG} = \vec{b} + \beta\left(-\vec{b} + \tfrac{1}{2}\vec{a}\right)$$

$\alpha = \frac{1}{2}\beta$

$\alpha = 1 - \beta$ $\qquad \frac{1}{2}\beta = 1 - \beta \qquad \frac{3}{2}\beta = 1 \qquad \beta = \frac{2}{3} \qquad \alpha = \frac{1}{3}$

$$\overrightarrow{AG} = \frac{1}{3}(\vec{a} + \vec{b})$$

$$\overrightarrow{HC} = \alpha'(\vec{a} + \vec{b}) = \beta'\left(\tfrac{1}{2}\vec{b} - \vec{a}\right) + \vec{a}$$

$$\overrightarrow{HC} = \frac{1}{3}(\vec{a} + \vec{b}) \qquad \overrightarrow{DF} = \vec{a} - \frac{1}{2}\vec{b}$$

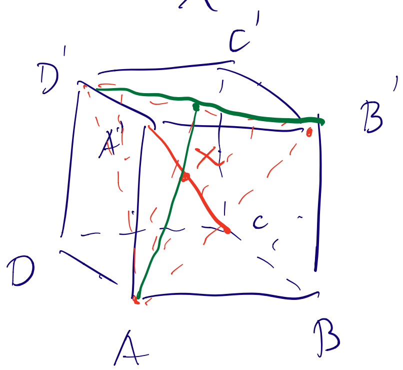
$\alpha' = 1 - \beta'$ $\quad \rightsquigarrow \quad \alpha' = \frac{1}{3}$

$\alpha' = \frac{1}{2}\beta'$ $\qquad \beta' = \frac{2}{3}$

$\overrightarrow{GH} = ?$

$$\overrightarrow{AG} + \overrightarrow{GH} + \overrightarrow{HC} = \overrightarrow{AC} = \frac{2}{3}\overrightarrow{AC} + \overrightarrow{GH} \rightsquigarrow \overrightarrow{GH} = \frac{1}{3}(\vec{a} + \vec{b})$$

5. V ~~kocki~~ romboid $ABCDA'B'C'D'$ izrazi presečišče trikotnika $AB'D'$ in diagonale $A'C$.



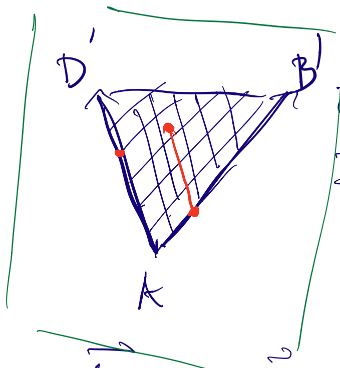
$\vec{AB} = \vec{a}$

$\vec{AD} = \vec{b}$

$\vec{AA'} = \vec{c}$

$\vec{AX}$

$$\vec{AX} = \vec{c} + \alpha \vec{A'C} = \vec{c} + \alpha(-\vec{c} + \vec{a} + \vec{b}) = \beta(\vec{b} + \vec{c}) + \gamma(\vec{a} + \vec{c})$$



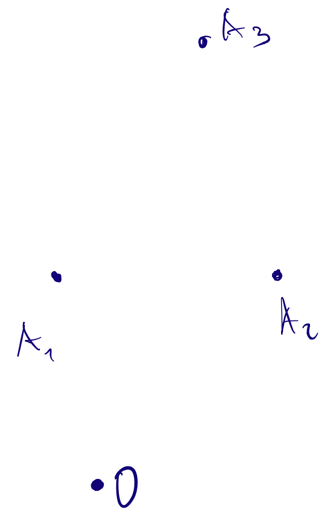
$\vec{c}$: $1 - \alpha = \beta + \gamma$ $\quad 1 - \alpha = 2\alpha \quad 1 = 3\alpha$

$\vec{a}$: $\alpha = \gamma$ $\quad \alpha = \frac{1}{3}$

$\vec{b}$: $\alpha = \beta$

$$\vec{AX} = \vec{c} - \frac{1}{3}\vec{c} + \frac{1}{3}\vec{a} + \frac{1}{3} b = \underline{\frac{1}{3}\vec{a} + \frac{1}{3}\vec{b} + \frac{2}{3}\vec{c}}$$

6. Naj bodo A_1, A_2 in A_3 poljubne točke ter a_1, a_2 in a_3 realna števila, za katera velja $a_1 +_2 a_2 + a_3 = 1$. Pokaži, da obstaja natanko ena točka A, za katero velja $a_1\overrightarrow{AA_1} + a_2\overrightarrow{AA_2} + a_3\overrightarrow{AA_3} = 0$ in $\overrightarrow{BA} = a_1\overrightarrow{BA_1} + a_2\overrightarrow{BA_2} + a_3\overrightarrow{BA_3}$ za katerokoli točko B.



$a_1 + a_2 + a_3 = 1$

$$\overrightarrow{AA_1} = \vec{r}_1 - \vec{r}_A$$

$$\overrightarrow{AA_2} = \vec{r}_2 - \vec{r}_A$$

$$\overrightarrow{AA_3} = \vec{r}_3 - r_A$$

$$a_1(\vec{r}_1 - \vec{r}_A) + a_2(\vec{r}_2 - \vec{r}_A) + a_3(\vec{r}_3 - \vec{r}_A) = 0$$

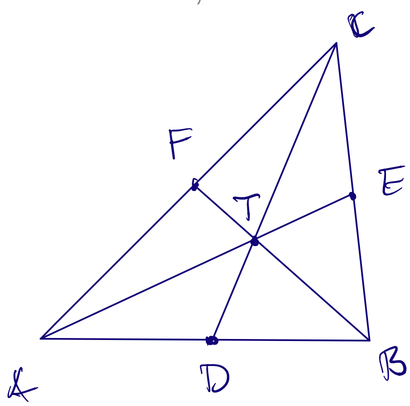
$$\underline{a_1\vec{r}_1 + a_2\vec{r}_2 + a_3\vec{r}_3} = (a_1 + a_2 + a_3)\vec{r}_A = \underline{\vec{r}_A}$$

$$A = a_1A_1 + a_2A_2 + a_3A_3$$

$$\overrightarrow{BA} = (\vec{r}_B - \vec{r}_A) = r_B - a_1\vec{r}_1 - a_2\vec{r}_2 - a_3\vec{r}_3 = a_1\overbrace{(\vec{r}_B - \vec{r}_1)}^{\overrightarrow{BA_1}} + a_2\overbrace{(\vec{r}_B - \vec{r}_2)}^{\overrightarrow{BA_2}} + a_3\overbrace{(\vec{r}_B - \vec{r}_3)}^{\overrightarrow{BA_3}}$$

$$r_B = (a_1 + a_2 + a_3)r_B$$

7. Dokaži, da se težiščnice trikotnika sekajo v eni točki.



$\vec{AB} = \vec{a}$

$\vec{BC} = \vec{b}$

$AE \times DC$

$\vec{AT} = x\,\vec{AE} = x(a + \frac{1}{2}\vec{b}) = \frac{1}{2}\vec{a} + y(\frac{1}{2}\vec{a} + \vec{b})$

$x = \frac{1}{2} + \frac{1}{2}y \qquad x = \frac{1}{2} + \frac{1}{4}x$

$\frac{1}{2}x = y \qquad \frac{3}{4}x = \frac{1}{2} \qquad x = \frac{2}{3}$

$AE \times DC \qquad \vec{AT} = \frac{2}{3}\vec{a} + \frac{1}{3}\vec{b}$

$\parallel$

$AE \times FB$

$\vec{AT} = x(a + \frac{1}{2}\vec{b}) = \frac{1}{2}\vec{AC} + y\,\vec{FB}$

$\vec{AC} = \vec{a} + \vec{b} \qquad \vec{FB} = -\frac{1}{2}(\vec{a} + \vec{b}) + \vec{a}$

$x(\vec{a} + \frac{1}{2}\vec{b}) = \frac{1}{2}(a+b) + y(\frac{1}{2}\vec{a} - \frac{1}{2}\vec{b})$

$x = \frac{1}{2} + \frac{1}{2}y$

$\frac{1}{2}x = \frac{1}{2} - \frac{1}{2}y \qquad +$

$\frac{3}{2}x = 1$

$x = \frac{2}{3}$