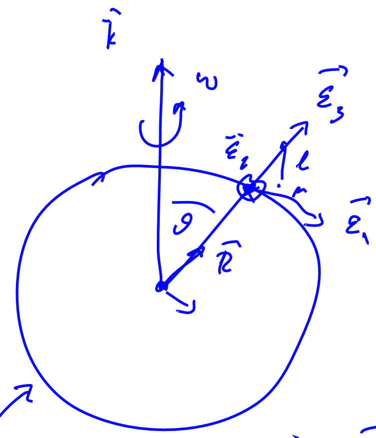
Predavanje 7. december 2020

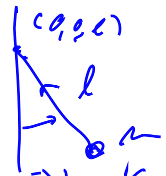
Foucaultovo nihalo



$$\Rightarrow m\vec a_{rel} = \vec F_{rel} = \vec F_g + \vec T - m\vec a_0 - \cancel{m\dot{\vec\omega}\times\vec\rho} - \cancel{m\vec\omega\times(\vec\omega\times\vec\rho)} - 2m\vec\omega\times\vec v_{rel}$$

$$\vec\omega = \omega\vec k; \quad \omega = \frac{2\pi}{24\cdot 3600\,s} \doteq 0.7\cdot 10^{-5} \qquad \dot{\vec\omega} = \vec 0$$

$$\vec a_0 = \vec\omega\times(\vec\omega\times\vec R) \qquad \vec a_0 = \vec\omega\times(\vec\omega\times\vec R)$$



$$\vec F_g - \vec a_0 = -mg\vec\varepsilon_3 \qquad \vec T = T\frac{1}{l}\left(-\rho_1\vec\varepsilon_1 - \rho_2\vec\varepsilon_2 + (l-\rho_3)\vec\varepsilon_3\right) \qquad (\rho_1,\rho_2,\rho_3)$$

$$\vec\omega = \omega\vec k = \omega(-\sin\vartheta\,\vec\varepsilon_1 + \cos\vartheta\,\vec\varepsilon_3)$$

$$\vec\omega\times\vec v_{rel} = \omega(-\sin\vartheta\,\vec\varepsilon_1 + \cos\vartheta\,\vec\varepsilon_3)\times(\dot\rho_1\vec\varepsilon_1 + \dot\rho_2\vec\varepsilon_2 + \dot\rho_3\vec\varepsilon_3) =$$

$$= \omega(-\dot\rho_2\sin\vartheta\,\vec\varepsilon_3 + \dot\rho_3\sin\vartheta\,\vec\varepsilon_2 + \dot\rho_1\cos\vartheta\,\vec\varepsilon_2 - \dot\rho_2\cos\vartheta\,\vec\varepsilon_1)$$

$$\left[\begin{aligned} m\ddot\rho_1 &= -\frac{T}{l}\rho_1 + 2m\omega\dot\rho_2\cos\vartheta \\ m\ddot\rho_2 &= -\frac{T}{l}\rho_2 - 2m\omega\dot\rho_3\sin\vartheta - 2m\omega\cos\vartheta\,\dot\rho_1 \\ m\ddot\rho_3 &= -mg + \frac{T}{l}(l-\rho_3) + 2m\omega\sin\vartheta\,\dot\rho_2 \end{aligned}\right. \qquad \left|\rho_1^2 + \rho_2^2 + (l-\rho_3)^2 = l^2\right|$$

$$\rho_3 \doteq 0; \quad \dot\rho_3 \doteq 0, \quad \ddot\rho_3 \doteq 0$$

$$\frac{l-\rho_3}{l} \doteq 1 \qquad 0 = -mg + T + 2m\omega\dot\rho_2\sin\vartheta$$

$$T = mg - 2m\omega\dot\rho_2\sin\vartheta$$

$$\left[\begin{aligned} m\ddot\rho_1 &= -\frac{1}{l}\rho_1(mg - 2m\omega\dot\rho_2\sin\vartheta) + 2m\omega\dot\rho_2\cos\vartheta \\ m\ddot\rho_2 &= -\frac{1}{l}\rho_2(mg - 2m\omega\dot\rho_2\sin\vartheta) - 2m\omega\dot\rho_1\cos\vartheta \end{aligned}\right] \qquad \boxed{\begin{aligned}\rho_1\dot\rho_2 &\doteq 0 \\ \rho_2\dot\rho_2 &\doteq 0\end{aligned}}$$

$$\ddot\rho_1 = -\frac{g}{l}\rho_1 + 2\omega\cos\vartheta\,\dot\rho_2 \qquad \frac{g}{l} = k^2 \qquad \omega\cos\vartheta = \beta$$

$$\ddot\rho_2 = -\frac{g}{l}\rho_2 - 2\omega\cos\vartheta\,\dot\rho_1 \quad / i \qquad |\beta| << 1$$

$z = \xi_1 + i\xi_2$ $\qquad \ddot{z} = -k^2 z + 2p(\dot{\xi}_2 - i\dot{\xi}_1)$

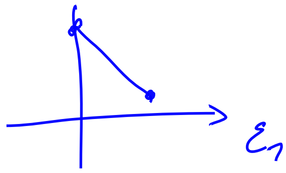
$-i\dot{z} = -i(\dot{\xi}_1 + i\dot{\xi}_2)$

$$\boxed{\ddot{z} + 2ip\dot{z} + k^2 z = 0}$$

$\lambda^2 + 2ip\lambda + k^2 = 0 \qquad \lambda_{1,2} = -ip \pm \sqrt{-p^2 - k^2} = i(-p \pm \underbrace{\sqrt{p^2+k^2}}_{k})$

$\lambda_{1,2} = i(-p \pm k)$

$z = (A_1 + iA_2)e^{-i(p+k)t} + (B_1 + iB_2)e^{-i(p-k)t}$



$z(0) = A_0 \in \mathbb{R} \qquad \dot{z}(0) = 0$

$A_1 + B_1 = A_0$

$A_2 + B_2 = 0$

$(p+k)A_1 + (p-k)B_1 = 0$ $\qquad (p-k) - (p+k) = -2k \neq 0$

$(p+k)A_2 + (p-k)B_2 = 0$ $\qquad A_2 = B_2 = 0$

$A_1 - B_1 = 0 \quad \Rightarrow \quad A_1 = B_1 = \frac{1}{2}A_0$

$\cos pt - i\sin pt$

$z = \frac{1}{2}A_0\left(e^{-i(p+k)t} + e^{-i(p-k)t}\right) =$

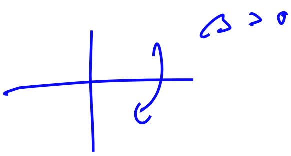
$= \frac{1}{2}A_0 e^{-ipt}\left(e^{-ikt} + e^{ikt}\right) = A_0 e^{-ipt}\cos kt$

$\xi_1 = A_0 \cos kt \cos pt = A_0 \cos\sqrt{\frac{g}{l}}\,t \cos(\omega\cos\vartheta\, t)$

$\xi_2 = -A_0 \cos kt \sin pt = -A_0 \cos\sqrt{\frac{g}{l}}\,t \sin(\omega\cos\vartheta\, t)$

$\vec{e}_1 = \cos(pt)\,\vec{\varepsilon}_1 - \sin(pt)\,\varepsilon_2$

$\vec{\xi} = A_0 \cos\sqrt{\frac{g}{l}}\,t\ \vec{e}_1$



$p = \omega\cos\vartheta > 0 \qquad 0 < \vartheta < \frac{\pi}{2}$

$\xi_2 \cdot \dot{\xi}_2 = \left(\sqrt{\frac{g}{l}}A \sin\sqrt{\frac{g}{l}}t \sin pt - A_0 \cos\sqrt{\frac{g}{l}}t\, p\cos pt\right)\xi_2$

$\frac{\omega}{\Omega}$

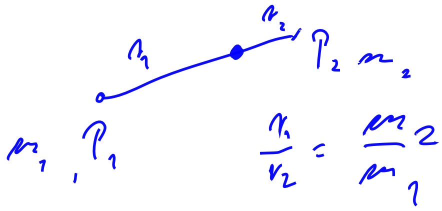
P_i ; $i=1,\dots,N$

m_i ; $\vec r_i = \vec{OP_i} = P_i - O$

Sistem materialnih točk

$$\vec r_* = \frac{1}{m}\sum_{i=1}^N m_i \vec r_i \quad ; \quad m = \sum_{i=1}^N m_i$$

Masno središče



$$\frac{r_1}{r_2} = \frac{m_2}{m_1}$$

$$m_i \ddot{\vec r}_i = \dot{\vec p}_i = \vec F_i + \sum_{\substack{j=1 \\ j\neq i}}^N \vec F_{ji}$$

sila P_j na P_i

$$\vec F_{ji} = \vec F_{ji}(P_j, P_i)$$

$$\boxed{\vec F_{ji} = -\vec F_{ij}}$$

$$\vec F_{ii} = \vec 0$$

Enačba gibanja masnega središča

Notranje sile, zunanje sile

$$\vec v_* = \dot{\vec r}_* = \frac{1}{m}\sum m_i \dot{\vec r}_i$$

$$\vec a_* = \frac{1}{m}\sum m_i \ddot{\vec r}_i$$

$$\sum_{i=1}^N m_i \ddot{\vec r}_i = \sum_{i=1}^N \vec F_i + \sum_{i,j=1}^N \vec F_{ji}$$

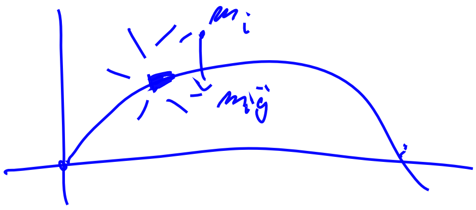
$$m\vec a_* = \vec F + \underbrace{\sum_{i,j=1}^N \vec F_{ji}}_{=\vec 0}$$

$$\boxed{m\vec a_* = \vec F}$$

Enačba gibanja masnega središča

$$m\vec{a}_* = \vec{F}.$$

Primer: poševni met.



$\vec{F}_i = m_i \cdot \vec{g}$ $\qquad \vec{F} = \sum_{i=1}^{N} \vec{F}_i = \sum_{i=1}^{N} m_i \vec{g} = m\vec{g}$

$\vec{F} = \sum_{i=1}^{N} \vec{F}_i \quad ; \quad \vec{F}_i = \vec{F}_i(P_i)$

odvisna od položaja P_i

Vrtilna količina $\quad \vec{l}_i = \vec{l}_i(O) = \vec{OP_i} \times m_i \vec{v}_i = (P_i - O) \times m_i \dot{P}_i$

$\vec{L} = \vec{L}(O) = \sum_{i=1}^{N} \vec{l}_i(O)$

Vrtilna količina točke $\vec{l}(O, P_i) = (P_i - O) \times m_i \dot{P}_i$, vrtilna količina sistema

$$\vec{L} = \vec{L}(O) = \sum_{i=1}^{N} \vec{l}(O, P_i).$$

$\vec{OP_i} \times$ **Izrek o vrtilni količini** / $m_i \ddot{\vec r}_i = \vec F_i + \sum_{j=1}^{N} \vec F_{ji}$

rezultanta navora notranjih sil

$\vec N_n(O)$

$$\sum_{i=1}^{N} \vec{OP_i} \times m_i \ddot{\vec r}_i = \sum_{i=1}^{N} \vec{OP_i} \times \vec F_i + \sum_{i,j=1}^{N} \vec{OP_i} \times \vec F_{ji}$$

$\vec{OP_i} = \vec r_i \;\Big|\; \vec r_i \times \vec F_i$

$$\sum_{i=1}^{N} \vec{OP_i} \times \vec F_i = \vec N(O)$$

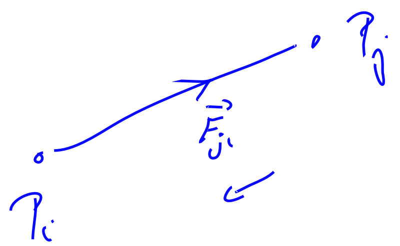
rezultanta navora zunanjih sil

$$\frac{d}{dt}\vec L = \sum_{i=1}^{N} \frac{d}{dt}\left(\vec{OP_i} \times m_i \dot{\vec r}_i\right) = \sum_{i=1}^{N} \vec{OP_i} \times m_i \ddot{\vec r}_i$$

$$\dot{\vec r}_i \times m_i \dot{\vec r}_i + \vec{OP_i} \times m_i \ddot{\vec r}_i$$

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\vec L = \vec N + \vec N_n.}$$

Izrek o vrtilni količini



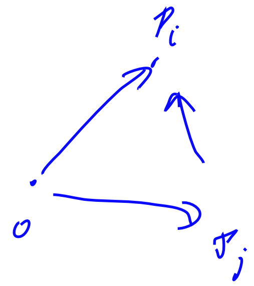
Definicija 1. *Sila F_{ji} je centralna, če je*

$$\vec F_{ji}(P_i, P_j) = F_{ji}(|P_i - P_j|)\frac{P_i - P_j}{|P_i - P_j|}, \quad P_i - P_j = \overrightarrow{P_jP_i}.$$

Navor notranji sil.

$$\vec N_n(O) = \sum_{i,j=1}^{n} (P_i - O)\times \vec F_{ji} = \frac{1}{2}\Big(\sum_{i,j=1}^{n}(P_i - O)\times \vec F_{ji} + \underbrace{\sum_{i,j=1}^{n}(P_i - O)\times \vec F_{ji}}\Big)$$

$$\sum_{i,j=1}^{n}(P_i - O)\times \vec F_{ji} = \sum_{j,i=1}^{n}(P_j - O)\times \vec F_{ij} = -\sum_{j,i=1}^{n}(P_j - O)\times \vec F_{ji}, \quad i,j=1$$



$$= \frac{1}{2}\sum_{i,j=1}^{n}\big(\underbrace{(P_i - O) - (P_j - O)}_{\vec{OP_i} - \vec{OP_j} = \vec{P_jP_i}}\big)\times \vec F_{ji} = \frac{1}{2}\sum_{i,j=1}^{n}\vec{P_jP_i}\times \vec F_{ji} = \vec 0$$

Izrek 1 (Izrek o vrtilni količini)**.** *Če so notraje sile centralne, je* $\frac{\mathrm{d}}{\mathrm{d}t}\vec L = \vec N$.

Odvisnost vrtilne količine od izbire pola

Trditev 1. *Za poljubno točko* P_0 *je* $\vec L(P_0) = \vec L(O) + \vec{P_0O}\times m\vec v_*$.

Izrek o vrtilni količin

Izrek 2 (Izrek o vrtilni količini)**.** $\frac{\mathrm{d}}{\mathrm{d}t}\vec L(P_*) = \vec N(P_*)$.

Vrtilna količina relativnega gibanja

Pisava: $\vec{r}_i = \vec{r}_0 + \vec{\zeta}_i$.

Definicija 2. *Vrtilna količina relativnega gibanja okoli P_0 je*

$$\vec{\mathcal{L}} = \vec{\mathcal{L}}(P_0) = \sum_i \vec{\zeta}_i \times m_i \dot{\vec{\zeta}}_i.$$

Trditev 2.

$$\vec{\mathcal{L}}(P_0) = \vec{L}(P_0) - m(\vec{r}_* - \vec{r}_0) \times \vec{v}_0.$$

Posledica 1.

$$\vec{L}(P_*) = \vec{\mathcal{L}}(P_*) \quad \textit{in} \quad \frac{\mathrm{d}}{\mathrm{d}t}\vec{\mathcal{L}}(P_*) = \vec{N}(P_*).$$

Kinetična enargija

Dekompozicija kinetične energije.