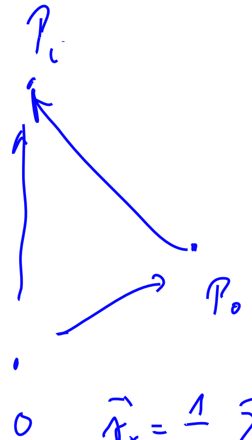


$\vec r_* = \frac{1}{m}\sum m_i \vec r_i$

Predavanje 14. december 2020

Odvisnost vrtilne količine od izbire pola

Trditev 1. *Za poljubno točko P_0 je $\vec L(P_0) = \vec L(O) + \vec{P_0O} \times m\vec v_*$.*

$$\vec L(O) = \sum_{i=1}^N (P_i - O)\times m_i \vec v_i = \sum_{i=1}^n \vec{OP_i}\times m_i \vec v_i$$

$$\vec L(P_0) = \sum_{i=1}^N \vec{P_0P_i}\times m_i\vec v_i = \underbrace{\sum_{i=1}^N \vec{OP_i}\times m_i\vec v_i}_{\vec L(O)} - \sum_{i=1}^n \vec{OP_0}\times m_i\vec v_i$$

$$\vec{P_0P_i} = \vec{OP_i} - \vec{OP_0}$$

$$\sum_{i=1}^n \vec{OP_0}\times m_i\vec v_i = \vec{OP_0}\times \underbrace{\sum_{i=1}^n m_i\vec v_i}_{m\vec v_*}$$

$$\vec L(P_0) = \vec L(O) - \vec{OP_0}\times m\vec v_*$$

Izrek o vrtilni količin

Izrek 1 (Izrek o vrtilni količini). $\frac{\mathrm{d}}{\mathrm{d}t}\vec L(P_*) = \vec N(P_*)$.

$$\frac{d\vec L(O)}{dt} = \vec N(O)$$

$$\vec L(P_*) = \vec L(O) - \underbrace{\vec{OP_*}}_{\vec r_*}\times m\vec v_*$$

$$\frac{d\vec L}{dt}(P_*) = \vec N(O) - \vec v_*\times m\vec v_* - \vec{OP_*}\times m\vec a_*$$

Vrtilna količina relativnega gibanja

Pisava: $\vec r_i = \vec r_0 + \vec\zeta_i$.

Definicija 1. *Vrtilna količina relativnega gibanja okoli P_0 je*

$$\vec{\mathcal L} = \vec{\mathcal L}(P_0) = \sum_i \vec\zeta_i \times m_i \dot{\vec\zeta}_i.$$

$$\frac{d\vec L}{dt}(P_*) = \sum_{i=1}^N \vec{OP_i}\times\vec F_i - \vec{OP_*}\times\sum_{i=1}^n \vec F_i = \sum_{i=1}^N (\vec{OP_i} - \vec{OP_*})\times \vec F_i$$

$$m\vec a_* = \vec F = \sum_{i=1}^N \vec F_i$$

$\vec{P_*P_i}$, $\vec\zeta_i$

$$\frac{d\vec L}{dt}(P_*) = \sum_{i=1}^n \vec{P_*P_i}\times \vec F_i = \vec N(P_*)$$

$(P_i - P_*)$

$$\vec L(P_0) = \sum_{i=1}^N \vec{P_0P_i}\times m_i\vec v_i$$

$$\vec{OP_i} = \vec r_i = \vec r_0 + \vec\zeta_i \qquad \vec r_0 = \vec{OP_0} \qquad \vec\zeta_i = \vec{P_0P_i}$$

Trditev 2.

$$\vec{\mathcal L}(P_0) = \vec L(P_0) - m(\vec r_* - \vec r_0)\times \vec v_0.$$

$$\vec{\mathcal L}(P_0) = \sum_{i=1}^N \vec\zeta_i\times m_i\dot{\vec\zeta}_i = \sum_{i=1}^N \vec\zeta_i \times m_i(\dot{\vec r}_i - \dot{\vec r}_0) = \vec L(P_0) - \sum_{i=1}^N m_i\vec\zeta_i\times\vec v_0$$

$\dot{\vec r}_i = \vec v_i$, $\dot{\vec r}_0 = \vec v_0$

$$\sum_{i=1}^{N} m_i \vec{\rho}_i = \sum_{i=1}^{N} m_i (\vec{r}_i - \vec{r}_0) = m\vec{r}_* - m\vec{r}_0$$

$$\vec{L}(P_0) = \vec{\mathcal{L}}(P_0) - (\vec{r}_* - \vec{r}_0) \times m\vec{v}_0$$

$$\vec{L}(P_0) \stackrel{?}{=} \vec{\mathcal{L}}(P_0) \quad \text{velja, če je} \quad P_0 = P_* \quad \vec{v}_0 = \vec{0}$$

$$\vec{L}(P_*) = \vec{\mathcal{L}}(P_*)$$

$$\frac{d\vec{\mathcal{L}}(P_*)}{dt} = \vec{N}(P_*)$$

Posledica 1.

$$\vec{L}(P_*) = \vec{\mathcal{L}}(P_*) \quad \textit{in} \quad \frac{\mathrm{d}}{\mathrm{d}t}\vec{\mathcal{L}}(P_*) = \vec{N}(P_*).$$

Kinetična energija

Dekompozicija kinetične energije.

$$\vec{r}_i = \vec{r}_0 + \vec{\rho}_i \Rightarrow \vec{v}_i = \vec{v}_0 + \dot{\vec{\rho}}_i$$

$$T = \frac{1}{2}\sum_{i=1}^{N} m_i |\vec{v}_i|^2 = \frac{1}{2}\sum_{i=1}^{N} m_i |\vec{v}_0|^2 + \frac{1}{2}\sum_{i=1}^{N} 2 m_i \vec{v}_0 \cdot \dot{\vec{\rho}}_i + \frac{1}{2}\sum_{i=1}^{N} m_i |\dot{\vec{\rho}}_i|^2$$

$$= \frac{1}{2} m |\vec{v}_0|^2 + \sum_{i=1}^{N} m_i \dot{\vec{\rho}}_i \cdot \vec{v}_0 + \mathcal{T}(P_0)$$

kinetična energija relativnega gibanja $\mathcal{T}(P_0)$

$$\dot{\vec{\rho}}_i = \dot{\vec{r}}_i - \dot{\vec{r}}_0$$

$$= \frac{1}{2} m |\vec{v}_0|^2 + m\vec{v}_* \cdot \vec{v}_0 - m|\vec{v}_0|^2 + \mathcal{T}(P_0)$$

$$T = \mathcal{T}(P_0) + m\vec{v}_* \cdot \vec{v}_0 - \frac{1}{2} m |\vec{v}_0|^2$$

$$P_0 = P_* \qquad T = \frac{1}{2} m |\vec{v}_*|^2 + \mathcal{T}(P_*) = \frac{1}{2} m |\vec{v}_*|^2 + T_*$$

Energijska enačba

$$T + U + \frac{1}{2}\sum_{i,j=1}^{n} U_{ji} = E_0.$$

$$m_i \ddot{\vec{r}}_i = \vec{F}_i + \sum_{\substack{j=1 \\ j \neq i}}^{N} \vec{F}_{ji} \quad / \cdot \dot{\vec{r}}_i$$

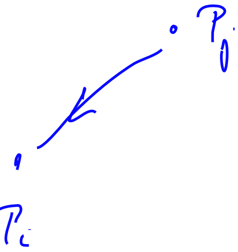
$$\frac{d}{dt}\left(\frac{1}{2} m_i \dot{\vec{r}}_i \cdot \dot{\vec{r}}_i\right) = \vec{F}_i \cdot \dot{\vec{r}}_i + \sum_{j=1}^{N} \vec{F}_{ji} \cdot \dot{\vec{r}}_i$$

$$\vec F_i = -\operatorname{grad} U_i(\vec r_i) \qquad \frac{dU_i}{dt} = \frac{d}{dt} U_i(\vec r_i(t)) = \operatorname{grad} U_i \cdot \dot{\vec r}_i$$

$$\frac{d}{dt}\left(\frac{1}{2} m_i |\vec v_i|^2 + U_i\right) = \sum_{i=1}^{N} \vec F_{ji} \cdot \dot{\vec r}_i \qquad \sum_{i=1}^{N}$$

$$\sum_{i=1}^{N} U_i = U$$

$$\frac{d}{dt}(T+U) = \underbrace{\sum_{i=1}^{N} \vec F_{ji} \cdot \dot{\vec r}_i}$$



$$\vec F_{ji} = F_{ji}\left(\underline{|\vec r_i - \vec r_j|}\right)\frac{\vec r_i - \vec r_j}{|\vec r_i - \vec r_j|}$$

$$\underline{U_{ji}} = -\int_{\rho_0}^{|\vec r_i - \vec r_j|} F_{ji}(\rho)\,d\rho + U_{ji}(\rho_0)$$

F_{ji} zvezne funkcije

$$\operatorname{grad}_{\vec r_i} U_{ji} = -F_{ji}(|\vec r_i - \vec r_j|)\operatorname{grad}_{\vec r_i}|\vec r_i - \vec r_j| = -F_{ji}(|\vec r_i - \vec r_j|)\frac{\vec r_i - \vec r_j}{|\vec r_i - \vec r_j|}$$

$$\operatorname{grad}_{\vec r} |\vec r| = \frac{\vec r}{|\vec r|} \qquad \underline{\vec F_{ji} = -\operatorname{grad}_{\vec r_i} U_{ji}} \ \checkmark$$

$$\vec F_{ji} = -\vec F_{ij} \qquad F_{ji} = F_{ij} \qquad U_{ji} = U_{ij}$$

$$U_{ji}(\vec r_i, \vec r_j) = U_{ji}(\vec r_j, \vec r_i) = U_{ij}(\vec r_j, \vec r_j)$$

$$\sum_{i \neq j} \vec F_{ji} \cdot \dot{\vec r}_i$$

$$\frac{dU_{ij}}{dt} = \frac{d}{dt} U_{ij}(\vec r_i, \vec r_j) = \operatorname{grad}_{\vec r_i} U_{ij} \cdot \dot{\vec r}_i + \operatorname{grad}_{\vec r_j} U_{ij} \cdot \dot{\vec r}_j =$$

$$= -\vec F_{ji} \cdot \dot{\vec r}_i - \vec F_{ij} \cdot \dot{\vec r}_j$$

$$\sum_{i \neq j} \underline{\vec F_{ji}} \cdot \dot{\vec r}_i = \frac{1}{2}\left(\sum_{i \neq j} \vec F_{ji} \dot{\vec r}_i + \sum_{j \neq i} \vec F_{ij} \dot{\vec r}_j\right) = -\frac{1}{2}\sum_{i \neq j} \frac{dU_{ij}}{dt}$$

$$U_n = \frac{1}{2}\sum_{i\neq j} U_{ij} = \sum_{1\le i<j\le N} U_{ij}$$

$$\frac{d}{dt}(T + U + U_n) = 0 \qquad \boxed{T + U + U_n = E_0}$$

Zaprti sistem

$$\vec{F}_i = 0 \quad \text{za} \quad \forall i$$

$$m\vec{a}_* = \vec{0} \Rightarrow m\vec{v}_* = \text{konst}$$

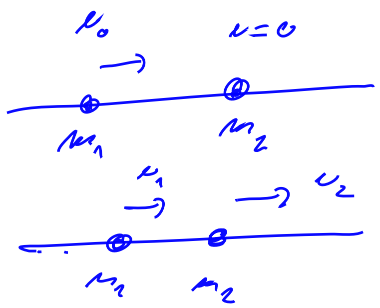
$$\sum m_i \vec{v}_i = \text{konst} = \vec{p}_0 \qquad m\vec{r}_* = \vec{p}_0 t + \vec{r}_*(t=0)$$

Izrek o ohranitvi gibalne količine.

$$\frac{d\vec{L}(0)}{dt} = \vec{0} \Rightarrow \vec{L}(0) = \text{konst.}$$

Izrek o ohranitvi vrtilne količine.

Trki



pred trkom

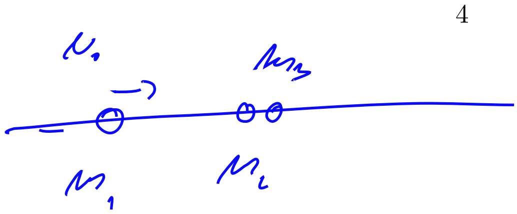
po trku

$$\begin{cases} m_1 v_0 = m_1 v_1 + m_2 v_2 \\ \frac{1}{2} m v_0^2 = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 \end{cases} \qquad \text{elastični trk}$$

$$v_2 \ge v_1$$

plastični trk; $v_1 = v_2$

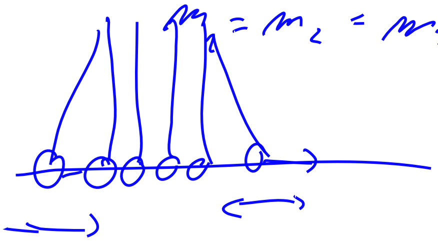
$$\frac{1}{2} m v_0^2 = \varepsilon\left(\frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2\right) \qquad 0 \le \varepsilon \le 1$$



$$m_1 v_0 = m_1 v_1 + m_2 v_2 + m_3 v_3 \quad ✓$$

$$v_1 \le v_2 \le v_3$$

$m_1 = m_2 = m_3 = m$



$v_1 = 0, \quad v_2 = 0, \quad v_3 = v_0$

$$\boxed{v_1 = -\frac{1}{3} v_0, \quad v_2 = \frac{2}{3} v_0, \quad v_3 = \frac{2}{3} v_0}$$

$$\frac{1}{2} m v_0^2 = \frac{1}{2} m \left(\frac{1}{9} + \frac{4}{9} + \frac{4}{9} \right) v_0^2 = \frac{1}{2} m v_0^2$$

Togo telo

Kinematika togega gibanja

Opis gibanja togega telesa, togo gibanje je natanko določeno z gibanjem treh nekolinearnih točk.

Opis togega gibanja kot relativno gibanje telesnega KS.

Posplošitev sistema materialnih točk na kontinuum mnogo točk, pojem togega telesa, masa.

Dinamične enačbe togega telesa

Tenzorski produkt

Tenzorski produkt dveh vektorjev $\vec{a}$ in $\vec{b}$ je linearna preslikava $\vec{a}\otimes\vec{b}$ definirana z $\vec{r} \mapsto (\vec{a}\otimes\vec{b})\vec{r} = (\vec{b}\cdot\vec{r})\vec{a}$.

Trditev 3. *Velja*

- $(\underline{\underline{A}})\vec{a} \otimes \vec{b} = (\underline{\underline{A}}\vec{a}) \otimes \vec{b}$;
- $(\vec{a} \otimes \vec{b})\underline{\underline{A}} = \vec{a} \otimes (\underline{\underline{A}}^T\vec{b})$;
- $(\vec{a} \otimes \vec{b})^T = \vec{b} \otimes \vec{a}$.