Vaje 22. december 2020

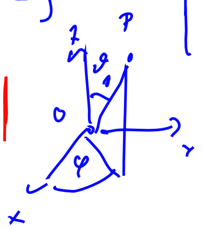
1. Izračunaj vztrajnostni tenzor elipsoida. $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\le 1$

$$J_{11}=\rho\int_B (y^2+z^2)\,dV;\quad J_{22}=\rho\int_B (x^2+z^2)\,dV;\quad J_{33}=\rho\int_B (x^2+y^2)\,dV$$

$$K_1=\rho\int_D x^2\,dV;\quad J_{11}=K_2+K_3,\quad J_{22}=K_1+K_3;\quad J_{33}=K_1+K_2$$

$$K_1=\rho\int_D x^2\,dV=\rho a^3bc\int_{B_1}\xi^2\,dV$$

$x=a\xi,\ y=b\eta,\ z=c\zeta$ $\quad\left|\frac{\partial(x,y,z)}{\partial(\xi,\eta,\zeta)}\right|=abc$



$\xi=r\cos\varphi\,\cancel{\cos}\sin\vartheta$

$\eta=r\sin\varphi\,\cancel{\cos}\sin\vartheta$

$\zeta=r\,\cancel{\sin}\cos\vartheta$

$$\left|\frac{\partial(\xi,\eta,\zeta)}{\partial(r,\vartheta,\varphi)}\right|=r^2\sin\vartheta$$

$$K_1=\rho a^3bc\int_0^1 r^2\,dr\int_0^\pi\sin\vartheta\,d\vartheta\int_0^{2\pi}d\varphi\ r^2\cos^2\varphi\,\cancel{\cos^2}\sin^2\vartheta=$$

$$=\rho a^3bc\int_0^1 r^4\,dr\ \cancel{\int_0^\pi\cos^2\vartheta\sin\vartheta\,d\vartheta}\int_0^{2\pi}\cos^2\varphi\,d\varphi=$$

$u=\cos\vartheta$

$du=-\sin\vartheta\,d\vartheta$

$\parallel\ \int_0^{2\pi}\sin^2\varphi\,d\varphi$

$$\int_0^\pi\sin^3\vartheta\,d\vartheta=\frac{4}{3}$$

$$=\rho a^3bc\ \frac{1}{5}\cdot\int_1^{-1}(-u^2)\,du\cdot\pi=$$

$$\rho a^3bc\cdot\frac{1}{5}\cdot\frac{4}{3}\cdot\pi$$

$$=\rho a^3bc\,\pi\,\frac{1}{5}\cdot 2\cdot\frac{1}{3}=\frac{2\cdot 2}{2\cdot 15}\rho a^3bc\,\pi$$

5·3

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$$m=\rho V(D)=\rho\cdot\frac{4}{3}\pi abc\qquad \cancel{K_1=a^2\frac{m}{10}}\qquad K_1=a^2\frac{1}{5}m$$

$$K_2 = \rho \int_{B} y^2 \, dV = \rho a b c^3 \int_{B_1} r^2 \sin^2\varphi \cos^2\vartheta \; r^2 \sin\vartheta \, dr \, d\vartheta \, d\varphi$$

$$K_2 = b^2 \frac{m}{10} \qquad \cos^2\vartheta \qquad K_2 = b^2 \frac{1}{5} m$$

$$K_3 = \rho \int_{B} z^2 dV = \rho a b c^3 \int_0^1 r^2 dr \int_0^\pi \sin\vartheta \, d\vartheta \int_0^{2\pi} d\varphi \cdot r^2 \cancel{\sin^2\vartheta} =$$

$$= \rho a b c^3 \frac{1}{5} \cdot 2\pi \int_0^\pi \sin^3\vartheta \, d\vartheta = \frac{2}{5}\pi \rho a b c^3 \left(\int_0^\pi \sin\vartheta \, d\vartheta - \int_0^\pi \sin\vartheta \cos^2\vartheta \, d\vartheta \right)$$

$$\sin^3\vartheta = \sin\vartheta (1 - \cos^2\vartheta) \, d\vartheta$$

$$= \frac{2}{5}\pi \rho a b c^3 \left(-\cos\vartheta \Big|_0^\pi + \frac{1}{3}\cos^3\vartheta \Big|_0^\pi \right) =$$

$$= \frac{2}{5}\pi \rho a b c^3 \underbrace{\left(2 + \frac{1}{3}(-1-1) \right)}_{2 - \frac{2}{3} = \frac{4}{3}} = \frac{8}{15}\pi \rho a b c^3 = \frac{2}{5} c^2 m$$

$$J_{33} = K_1 + K_2 = \frac{m}{10}(a^2 + b^2)$$

$$K_3 = \rho a b c^3 \cdot \frac{1}{5} \cdot 2\pi \cdot \frac{2}{3} = \rho a b c^3 \pi \frac{4}{15} = \frac{1}{5} c^2 m$$

$$J_{11} = \frac{1}{5} m (b^2 + c^2) \qquad J_{22} = \frac{1}{5} m (a^2 + c^2) \qquad J_{33} = \frac{1}{5} m (a^2 + b^2)$$

$$J_{12} = \rho a b^2 c^2 \int_0^1 r^2 dr \int_0^\pi \sin\vartheta \, d\vartheta \int_0^{2\pi} d\varphi \cdot r^2 \sin\varphi \cos\varphi \cdot \sin^2\vartheta$$

xy

$$\frac{1}{2} \int_0^{2\pi} \sin 2\varphi \, d\varphi = 0 \qquad J_{12} = 0$$

$\underline{\underline{J}}$ je diagonalna, Poseben primer $a = b = c$

$$\underline{\underline{J}} = \frac{2}{5} m a^2 \, \underline{\underline{I}}$$

$\vec{e}$ $\qquad J_e = \underline{\vec{e} \cdot \underline{\underline{J}}\, \vec{e}}$

2. Izračunaj vztrajnostni moment kocke okrog glavne diagonale.

$J_{11} = K_2 + K_3 \quad , \quad J_{22} = K_1 + K_3 \quad , \quad J_{33} = K_1 + K_2$

$$K_1 = \rho \int_B x^2 \, dV = \rho \int_{-a/2}^{a/2} x^2 dx \int_{-a/2}^{a/2} dy \int_{-a/2}^{a/2} dz = \rho a^2 \cdot 2 \cdot \frac{1}{3} \left(\frac{a}{2}\right)^3 = \rho a^5 \frac{1}{3 \cdot 4} =$$

$$= \frac{1}{12} \rho a^5 \quad \frac{1}{12} m a^2$$

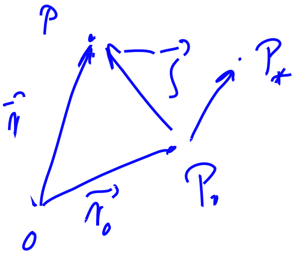
$K_1 = K_2 = K_3$

$J_{11} = J_{22} = J_{33} = \frac{1}{6} m a^2$

$D_{ij} = \rho \int_B x_i \cdot x_j \, dV \qquad J_{12} = \rho \int_B x y \, dV = \rho \int_{-a/2}^{a/2} x \, dx \int_{-a/2}^{a/2} y \, dy \int_{-a/2}^{a/2} dz = 0$

$\underline{\underline{J}} = \frac{1}{6} m a^2 \, \underline{\underline{I}} \qquad \vec{e} = \frac{1}{\sqrt{3}} (\vec{i} + \vec{j} + \vec{k})$

$$\vec{e} \cdot \underline{\underline{J}}\, \vec{e} = \frac{1}{6} m a^2 \frac{1}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \underline{\frac{1}{6} m a^2}$$



3. Steinerjev izrek.

$$\underline{\underline{J}}(P_0) = \int_B \left(|\vec{\rho}|^2 \underline{\underline{I}} - \vec{\rho} \otimes \vec{\rho} \right) dm$$

$$\overrightarrow{P_0 P} = P - P_0 = (P - P_*) + (P_* - P_0)$$

$$|\vec{\rho}|^2 = \vec{\rho} \cdot \vec{\rho} = \big((P - P_*) + (P_* - P_0) \big) \cdot \big((P - P_*) + (P_* - P_0) \big) =$$

$$|P - P_*|^2 + 2 (P - P_*) \cdot (P_* - P_0) + |P_* - P_0|^2$$

$$\vec{\rho} \otimes \vec{\rho} = (P - P_*) \otimes (P - P_*) + (P - P_*) \otimes (P_* - P_0) + (P_* - P_0) \otimes (P - P_*)$$
$$+ (P_* - P_0) \otimes (P_* - P_0)$$

$$\boxed{\underline{\underline{J}}(P_0) = \underline{\underline{J}}(P_*) + |P_* - P_0|^2 \underline{\underline{I}} m - m (P_* - P_0) \otimes (P_* - P_0)}$$

$$\int_B (P - P_*)\, dm = \int_B (\vec{r} - \vec{r}_*)\, dm = \int_B \vec{r}\, dm - \int_B \vec{r}_*\, dm = m \vec{r}_* - m \vec{r}_* = \vec{0}$$

Ravninski primer

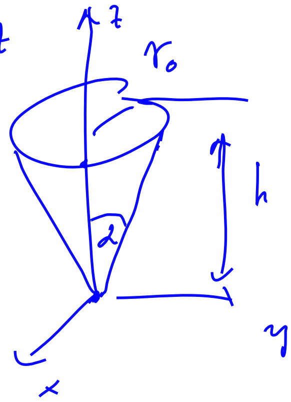
$$\vec{e} = \vec{n} \qquad J_0 = \vec{n} \cdot \underline{\underline{J}}(P_0)\, \vec{n} = J_* + |P_* - P_0|^2 m$$

P_* in P sta točki iz ravnine

$$\vec{n} \cdot (P_* - P_0) \otimes (P_* - P_0)\, \vec{n} = \left(\vec{n} \cdot (P_* - P_0) \right)^2 = 0$$

$$z_* = \frac{1}{m}\iiint_D z\,dV = \frac{\rho}{m}\int_0^{2\pi}d\varphi\int_0^{h}dz\int_0^{\frac{r_0}{h}z} r\,dr\cdot z = \frac{\rho}{m}2\pi\frac{1}{2}\int_0^h \frac{r_0^2}{h^2}z^3\,dz$$

$$= \frac{\rho}{m}\pi\frac{r_0^2}{h^2}\frac{1}{4}h^4 = \frac{3}{4}h$$



4. Izračunaj vztrajnostni tenzor stožca:

   (a) s polom v vrhu stožca; ✓

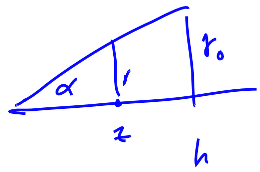
   (b) s polom v masnem središču; ✓

   (c) v koordinatnem sistemu z eno osjo v smeri tvorilke stožca.

$m = \rho V =$

$P_0 = (0,0,0)$

$P_* = (0,0,\frac{3}{4}h)$



$$V = \int_D dV = \int_0^{2\pi}d\varphi\int_0^h dz\int_0^{\frac{r_0}{h}z} r\,dr =$$

$$\operatorname{tg}\alpha = \frac{r_0}{h} = \frac{r}{z}$$

$$= 2\pi\int_0^h \frac{1}{2}\frac{r_0^2}{h^2}z^2\,dz = \pi\frac{r_0^2}{h^2}\cdot\frac{1}{3}h^3 = \frac{1}{3}\pi r_0^2 h$$

$$r = \frac{r_0}{h}z$$

$$K_1 = \rho\int_D x^2\,dV = \rho\int_0^{2\pi}d\varphi\int_0^h dz\int_0^{\frac{r_0}{h}z} r\,dr\cdot r^2\cos^2\varphi = \rho\int_0^{2\pi}\cos^2\varphi\,d\varphi\int_0^h dz\int_0^{\frac{r_0}{h}z} r^3\,dr =$$

$$= \pi\frac{1}{4}\rho\int_0^h\frac{r_0^4}{h^4}\cdot z^4\,dz = \rho\frac{1}{4}\pi\frac{r_0^4}{h^4}\cdot\frac{1}{5}h^5 = \rho\frac{1}{20}\cdot\pi r_0^4\cdot h = \frac{3}{20}\cdot r_0^2 m$$

$$K_2 = \rho\int_D y^2\,dV = \frac{3}{20}m r_0^2$$

$$K_3 = \rho\int_D z^2\,dV = \rho\int_0^{2\pi}d\varphi\int_0^h dz\int_0^{\frac{r_0}{h}z} r\,dr\cdot z^2 = \rho 2\pi\int_0^h\frac{1}{2}\frac{r_0^2}{h^2}\cdot z^4\,dz =$$

$$= \rho\pi\frac{r_0^2}{h^2}\cdot\frac{1}{5}h^5 = \frac{1}{5}\rho\pi r_0^2 h^3 = \frac{3}{5}m h^2$$

$$J_{11} = K_2 + K_3 = \frac{1}{5}m\left(3h^2 + \frac{3}{4}r_0^2\right) = J_{22}$$

$$J_{33} = K_1 + K_2 = \frac{3}{10}m r_0^2$$

$$J_{12} = -\rho\int_D xy\,dV = -\rho\underbrace{\int_0^{2\pi}\cos\varphi\sin\varphi}_{=0}\int_0^h dz\ldots$$