Predavanje 28. december 2020

Izrek 1 (Steiner)**.**

$$\underline{\underline{J}}(P_0) = \underline{\underline{J}}(P_*) + m|P_0 - P_*|^2\underline{\underline{I}} - m(P_0 - P_*) \otimes (P_0 - P_*).$$

Trditev 1 (Prehod na novo bazo)**.**

$$\vec{e}\,'i = {\alpha_i}^j \vec{e}_j \Longrightarrow J'_{ij} = {\alpha_i}^k {\alpha_j}^l J_{kl}.$$

$\vec{e}_j \qquad \vec{e}_i\,' = \alpha_{ij}\,\vec{e}_j$

$J'_{kl} = \vec{e}_k\,' \cdot \underline{\underline{J}}\,\vec{e}_l\,' = \alpha_{ki}\,\vec{e}_i \cdot \underline{\underline{J}}\,\alpha_{lj}\,\vec{e}_j =$

$= \alpha_{ki}\alpha_{lj}\,\vec{e}_i \cdot \underline{\underline{J}}\,\vec{e}_j = \alpha_{ki}\alpha_{lj}\,J_{ij}$

Simetrije vztranostnega tenzorja

Trditev 2. *Normala na ravnino zrcalne simetrije skozi P_0 je glavna smer vztrajnostnega tenzorja $\underline{\underline{J}}(P_0)$.*

Σ ; $P_0 \in \Sigma$; $\vec{e}$ normala na Σ

Z zrcaljenje preko Σ $\qquad Z(\mathcal{B}) = \mathcal{B}$

$\underline{\underline{J}}(P_0)\,\vec{e} \parallel \vec{e}$ $\qquad \underline{\underline{J}}(P_0) = \int_{\mathcal{B}} \left(|P-P_0|^2 \underline{\underline{I}} - (P-P_0)\otimes(P-P_0) \right) dm$

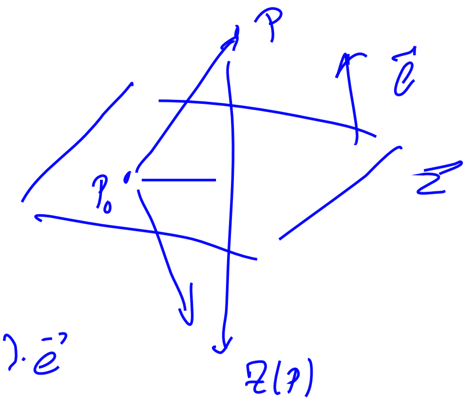
$\vec{\jmath} = P - P_0 \qquad\qquad dm = \rho\, dV$

$\underline{\underline{J}}(P_0)\,\vec{e}$ $\qquad \underline{\underline{D}}(P_0) = \int_{\mathcal{B}} (P-P_0)\otimes(P-P_0)\, dm$; $\quad \underline{\underline{D}}(P_0)\,\vec{e} \parallel \vec{e}$ ✓

$\vec{f} \perp \vec{e}$ $\qquad \underline{\underline{D}}(P_0)\,\vec{e}\cdot\vec{f} = \int_{\mathcal{B}} \left((P-P_0)\cdot\vec{e} \right)\left((P-P_0)\cdot\vec{f} \right) dm$

$$= \int \big((Z(P)-P_0)\cdot \vec{e}\big)\big((Z(P)-P_0)\cdot \vec{f}\big)\, dm$$

$$Z(D) = D$$



$$Z(P) - P_0 = (P-P_0) - 2\big((P-P_0)\cdot\vec{e}\big)\vec{e}$$

$$(Z(P)-P_0)\cdot\vec{e} = (P-P_0)\cdot\vec{e} - 2(P-P_0)\cdot\vec{e} = -(P-P_0)\cdot\vec{e}$$

$$(Z(P)-P_0)\cdot\vec{f} = (P-P_0)\cdot\vec{f}$$

$$D(P_0)\vec{e}\cdot\vec{f} = -\int_D \big((P-P_0)\cdot\vec{e}\big)\big((P-P_0)\cdot\vec{f}\big)\, dm = -D(P_0)\vec{e}\cdot\vec{f} \Rightarrow D(P_0)\vec{e}\cdot\vec{f} = 0$$

Trditev 3. *Če ima telo dve ravnini zrcalne simetrije skozi P_0, je presek ravnin simetrij glavna smer vztrajnostnega tenzorja $\underline{\underline{J}}(P_0)$.*

$\vec{e}_1$ in $\vec{e}_2$ normali na Σ_1, Σ_2

$\vec{e} = \vec{e}_1 \times \vec{e}_2$ $\qquad \underline{\underline{J}}(P_0)\vec{e}\cdot\vec{e}_i = 0 \quad i = 1,2$ ✓

$$\underline{\underline{J}}(P_0)\vec{e}\cdot\vec{e}_i = \vec{e}\cdot\underline{\underline{J}}(P_0)\vec{e}_i = \vec{e}\,\lambda_i \vec{e}_i = \lambda_i(\vec{e}_1\times\vec{e}_2)\cdot\vec{e}_i = 0$$

$$\underline{\underline{J}}(P_0)\vec{e} = \lambda_3\vec{e}$$

Kinetična energija togega telesa

$$T = \frac{1}{2}m|\vec{v}_*|^2 + \frac{1}{2}\vec{\omega}\cdot\underline{\underline{J}}(P_*)\vec{\omega}.$$

$$T_* = \frac{1}{2}\int_{\mathcal{B}} |\dot{\vec{\rho}}|^2 \, dm$$

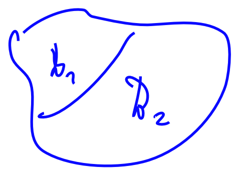
$$\vec{\rho} = P - P_* \; ; \quad \dot{\vec{\rho}} = \vec{v} - \vec{v}_* = \vec{\omega}\times\vec{\rho}$$

$$\vec{v} = \vec{v}_* + \vec{\omega}\times\vec{\rho} + \vec{0}$$

$$|\dot{\vec{\rho}}|^2 = \dot{\vec{\rho}}\cdot\dot{\vec{\rho}} = (\vec{\omega}\times\vec{\rho})(\vec{\omega}\times\vec{\rho}) = |\vec{\omega}|^2|\vec{\rho}|^2 - (\vec{\omega}\cdot\vec{\rho})^2 = \vec{\omega}\cdot\left(|\vec{\rho}|^2\underline{\underline{I}} - \vec{\rho}\otimes\vec{\rho}\right)\vec{\omega}$$

$$T_* = \frac{1}{2}\vec{\omega}\underbrace{\int_{\mathcal{B}}\left(|\vec{\rho}|^2\underline{\underline{I}} - \vec{\rho}\otimes\vec{\rho}\right)dm}\cdot\vec{\omega} = \frac{1}{2}\vec{\omega}\cdot\underline{\underline{J}}(P_*)\vec{\omega}$$

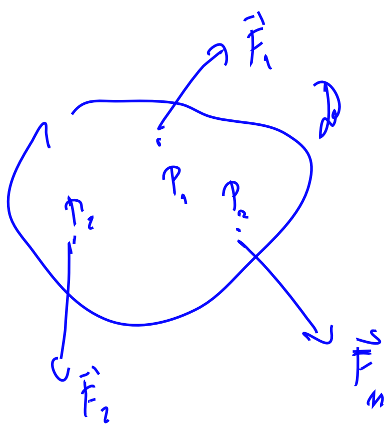
Dinamika togega telesa



Dinamične enačbe togega telesa so:

Princip

1. enačba gibanja masnega središča: $m\ddot{\vec{r}}_* = \vec{F}$;
2. ~~iz~~rek o vrtilni količini: $\frac{\mathrm{d}\vec{\mathcal{L}}(P_*)}{\mathrm{d}t} = \vec{N}(P_*)$;
3. $\vec{N}(P_0) = \vec{N}(P_*) + (P_* - P_0)\times\vec{F}$.



$$\mathcal{F} = \{(P_1,\vec{F}_1),\dots,(P_n,\vec{F}_n)\}$$

$$\vec{R}(\mathcal{F}) = \sum_{i=1}^{n}\vec{F}_i$$

$$\vec{N}(\mathcal{F},P_0) = \sum_{i=1}^{n}\vec{P_0P_i}\times\vec{F}_i$$

$\mathcal{F}' = \{(P_1', \vec{F}_1'), \dots, (P_m', \vec{F}_m')\}$; $\vec{R}(\mathcal{F}')$

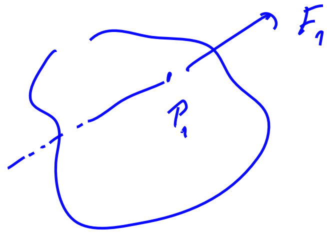
$\underline{\vec{R}(\mathcal{F}) = \vec{R}(\mathcal{F}')}$ $\quad$ $\underline{\vec{N}(\mathcal{F}, P_0) = \vec{N}(\mathcal{F}', P_0)}$ $\quad$ $\vec{N}(\mathcal{F}', P_0)$

Sistem sil

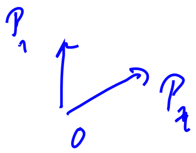
Sistem sil; točkovne sile, volumenske sile. Ekvipolentnost sistema sil. Dva ekvipolentna sistema sil

$\mathcal{F}$ in $\mathcal{F}'$ sta ekvipolentna sistema sil

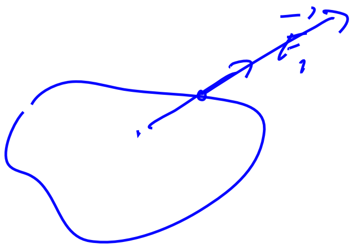
$$\vec{N}(\mathcal{F}, O) = \sum (P_i - O) \times \vec{F}_i = \sum (P_i - P_0) \times \vec{F}_i + (P_0 - O) \times \sum \vec{F}_i =$$
$$\vec{N}(\mathcal{F}, P_0) + (P_0 - O) \times \vec{R}(\mathcal{F}) =$$
$$\vec{N}(\mathcal{F}', P_0) + (P_0 - O) \times \vec{R}(\mathcal{F}') = \vec{N}(\mathcal{F}', O)$$



$$\{(P_1, \vec{F}_1)\} \equiv \{(P_1', \vec{F}_1)\}$$
$$(P_1 - O) \times \vec{F}_1 = (P_1' - O) \times \vec{F}_1$$
$$((P_1 - O) - (P_1' - O)) \times \vec{F}_1 = \vec{0}$$
$$(P_1 - P_1') \times \vec{F}_1 = \vec{0} \Rightarrow P_1 - P_1' \parallel \vec{F}_1$$



Polzna sila



$\mathcal{F}$ ima skupno prijemališče, če obstaja točka P_0 taka, da je $\vec{N}(\mathcal{F}, P_0) = \vec{0}$.

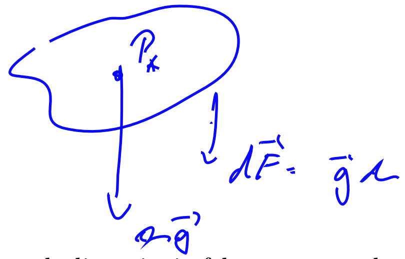
$\mathcal{F} = \{(P_1, \vec{F}), (P_2, -\vec{F})\}$ dvojica ; $\vec{N}(\mathcal{F}, P_1) \neq \vec{0}$

$\underline{\vec{R}(\mathcal{F}) = \vec{0}}$; $\vec{N}(\mathcal{F}, P_0) = (P_1 - P_0) \times \vec{F} - (P_2 - P_0) \times \vec{F} = \underline{(P_1 - P_2) \times \vec{F}}$

Volumenska sila

$\vec{F} = \int_D \vec{f}\, dV$; $\vec{f}$ gostota volumenske sile

Primer: sila teže $\vec{f} = \rho \vec{g}$;



imata enak dinamični efekt na togo telo.

Trditev 4. *Homogena sila je ekvipolentna svoji rezultanti s prijemališčem v masnem središču.*

"gostota sile je konstantna"

$$\vec F = \int_D \vec g \rho\, dV = \vec g \int_D dm = m\vec g\,;$$

$$\vec N(O) = \int_D (P-O)\times \vec f\, dV = \int_D \underbrace{(P-O)}_{\vec r}\, dm \times \vec g = \int_D \vec r\, dm \times \vec g = m\vec r_* \times \vec g = \vec r_* \times \vec F$$

$$\mathfrak{F} = \{(P_*, \vec F)\} \qquad \vec R(\mathfrak F) = \vec F\,; \quad \vec N(\mathfrak F, O) = (P_* - O)\times \vec F$$

$$\frac{d}{dt}\vec\Gamma(P_0) = \vec N(P_0) \qquad \vec\Gamma(P_0) = \underline{\underline J}(P_0)\vec\omega = J_{ij}(P_0)\,\omega_j\, \vec\varepsilon_i\,; \quad \vec\varepsilon_i = Q\vec e_i$$

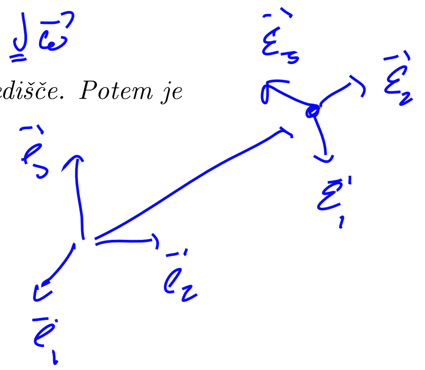
$$\dot{\vec\varepsilon}_i = \vec\omega \times \vec\varepsilon_i$$

$$\dot{\vec\Gamma}(P_0) = J_{ij}(P_0)\,\dot\omega_j\, \vec\varepsilon_i + J_{ij}(P_0)\,\omega_j\, \vec\omega\times\vec\varepsilon_i = \underline{\underline J}\,\dot{\vec\omega} + \vec\omega\times\underline{\underline J}\,\vec\omega$$

Eulerjeva dinamična enačba

Izrek 2 (Eulerjeva dinamična enačba)**.** *Naj bo P_0 fiksna točka ali masno središče. Potem je*

$$\underline{\underline{J}}(P_0)\dot{\vec\omega} + \vec\omega \times \underline{\underline{J}}(P_0)\vec\omega = \vec N(P_0).$$



Komponentni zapis Eulerjeve dinamične enačbe.

V lastnem koordinatnem sistemu

$$N_1 = J_1\dot\omega_1 - \omega_2\omega_3(J_2 - J_3),$$
$$N_2 = J_2\dot\omega_2 - \omega_3\omega_1(J_3 - J_1),$$
$$N_3 = J_3\dot\omega_3 - \omega_1\omega_2(J_1 - J_2).$$

$$\dot{\vec\omega} = \dot\omega_i\, \vec\varepsilon_i$$

$$\underline{\underline J}(P_0) = \begin{bmatrix} J_1 & & 0 \\ & J_2 & \\ 0 & & J_3 \end{bmatrix}$$

$$\underline{\underline J}(P_0)\,\dot{\vec\omega} = J_1\dot\omega_1\vec\varepsilon_1 + J_2\dot\omega_2\vec\varepsilon_2 + J_3\dot\omega_3\vec\varepsilon_3$$

$$\vec\omega \times \underline{\underline J}(P_0)\vec\omega = (\omega_1\vec\varepsilon_1 + \omega_2\vec\varepsilon_2 + \omega_3\vec\varepsilon_3)\times(J_1\omega_1\vec\varepsilon_1 + J_2\omega_2\vec\varepsilon_2 + J_3\omega_3\vec\varepsilon_3) =$$

$$= \omega_1\omega_2 J_2\vec\varepsilon_3 - \omega_1\omega_3 J_3\vec\varepsilon_2 - \omega_2 J_1\omega_1\vec\varepsilon_3 + \omega_2 J_3\omega_3\vec\varepsilon_1$$
$$+ \omega_3 J_1\omega_1\vec\varepsilon_2 - \omega_3 J_2\omega_2\vec\varepsilon_1 =$$

$$= \omega_2\omega_3(J_3 - J_2)\vec{\varepsilon}_1 + \omega_1\omega_3(J_1 - J_3)\vec{\varepsilon}_2 + \omega_1\omega_2(J_2 - J_1)\vec{\varepsilon}_3$$

$$\left.\begin{aligned}
J_1\dot{\omega}_1 - \omega_2\omega_3(J_2 - J_3) &= N_1\\
J_2\dot{\omega}_2 - \omega_3\omega_1(J_3 - J_1) &= N_2\\
J_3\dot{\omega}_3 - \omega_1\omega_2(J_1 - J_2) &= N_3
\end{aligned}\right|$$

V koordinatnem sistemu s koordinatno osjo v smeri stalne osi rotacije

$$\vec{\omega} = \dot{\varphi}\,\vec{\varepsilon}_3$$

$$\dot{\vec{\omega}} = \ddot{\varphi}\,\vec{\varepsilon}_3$$

$$J\dot{\vec{\omega}} + \vec{\omega}\times J\vec{\omega} = \vec{N}$$

$$\left[\quad\right]\begin{bmatrix}0\\0\\\ddot{\varphi}\end{bmatrix}$$

$$\dot{\varphi}\vec{\varepsilon}_3 \times \left(J_{13}\dot{\varphi}\vec{\varepsilon}_1 + J_{23}\dot{\varphi}\vec{\varepsilon}_2 + J_{33}\dot{\varphi}\vec{\varepsilon}_3\right) =$$

$$= \dot{\varphi}^2\left(J_{13}\vec{\varepsilon}_2 - J_{23}\vec{\varepsilon}_1\right)$$

$$\left.\begin{aligned}
J_{13}\ddot{\varphi} - J_{23}\dot{\varphi}^2 &= N_1\\
J_{23}\ddot{\varphi} + J_{13}\dot{\varphi}^2 &= N_2\\
\boxed{J_{33}\ddot{\varphi} = N_3}
\end{aligned}\right] \Rightarrow \varphi = \varphi(t)$$

$$N_1 = J_{13}\ddot{\varphi} - J_{23}\dot{\varphi}^2,$$
$$N_2 = J_{23}\ddot{\varphi} + J_{13}\dot{\varphi}^2,$$
$$N_3 = J_{33}\ddot{\varphi}$$

Izrek o delu

Trditev 5 (Izrek o delu).

$$\frac{\mathrm{d}T}{\mathrm{d}t} = \vec{v}_* \cdot \vec{F} + \vec{\omega} \cdot \vec{N}_*.$$

$$T = \frac{1}{2} m |\vec{v}_*|^2 + \frac{1}{2} \vec{\omega} \cdot J_* \vec{\omega}$$

$$\frac{dT}{dt} = m \vec{a}_* \cdot \vec{v}_* + \frac{1}{2} \dot{\vec{\omega}} \cdot J \vec{\omega} + \frac{1}{2} \vec{\omega} \cdot J \dot{\vec{\omega}} = \vec{v}_* \cdot \vec{F} + \vec{\omega} \cdot J \dot{\vec{\omega}}$$

$$J_* \dot{\vec{\omega}} + \vec{\omega} \times J_* \vec{\omega} = \vec{N}_* \qquad \vec{\omega} \cdot (\vec{\omega} \times J_* \vec{\omega}) = \vec{0}$$

$$= \vec{v}_* \cdot \vec{F} + \vec{\omega} \cdot \vec{N}_*$$

Izrek 3 (Izrek o energiji). *Če je sistem sil na togo telo konzervativen, je vsota kinetične in potencialne energije konstanta gibanja.*

Prosta vrtavka