Predavanje 4. januar 2021

Izrek 1 (Izrek o energiji)**.** *Če je sistem sil na togo telo konzervativen, je vsota kinetične in potencialne energije konstanta gibanja.*

$$\frac{dT}{dt} = \vec{F}\cdot\vec{v}_* + \vec{\omega}\cdot\vec{N}_*$$

$$\vec{F} = \int_{\mathcal{B}} \vec{f}\,dm \;:\quad \vec{f} = -\operatorname{grad} u \;;\quad U = \int_{\mathcal{B}} u(P)\,dm(P)$$

$$\frac{dU}{dt} = \frac{d}{dt}\int_{\mathcal{B}(t)} u\,dm = \frac{d}{dt}\int_{\mathcal{B}(t)} u(\vec{r})\,dm(\vec{r})$$

$$\vec{r} = \vec{r}_* + \vec{\rho} = \vec{r}_* + Q\vec{\xi}\;;\quad \vec{\xi}\in\mathcal{B}(t=0)$$

$$j = \det\frac{\partial\vec{r}}{\partial\vec{\xi}} = \det Q = 1$$

$$= \int_{\mathcal{B}(t=0)} \frac{d}{dt} u(\vec{r}_* + Q\vec{\xi})\,dm(\vec{\xi}) = \int_{\mathcal{B}(t=0)} \operatorname{grad} u\cdot(\vec{v}_* + \vec{\omega}\times\vec{\rho})\,dm(\vec{\xi}) =$$

$$= \int_{\mathcal{B}} \operatorname{grad} u\,dm\cdot\vec{v}_* - \int_{\mathcal{B}} \underbrace{\vec{f}\cdot(\vec{\omega}\times\vec{\rho})}_{\vec{\omega}\cdot(\vec{\rho}\times\vec{f})}\,dm = -\vec{F}\cdot\vec{v}_* - \vec{\omega}\cdot\underbrace{\int_{\mathcal{B}}\vec{\rho}\times\vec{f}\,dm}_{\vec{N}_*}$$

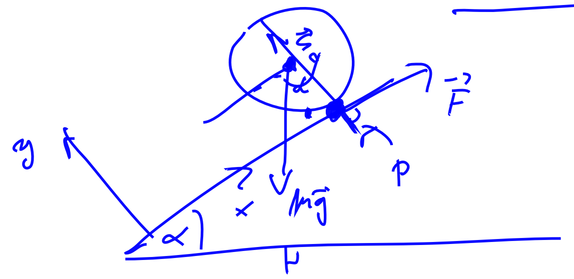
$$\frac{dU}{dt} = -\vec{F}\cdot\vec{v}_* - \vec{\omega}\cdot\vec{N}_* \;\Rightarrow\; \frac{dT}{dt} = -\frac{dU}{dt}\;;\quad \frac{d}{dt}(T+U) = 0$$

$$T+U = E_0$$

Primeri

Kotaljenje valja po strmini

- Brez drsenja.
- Kotaljenje z drsenjem.



$$m\vec{a}_* = m\vec{g} + \vec{S} + \vec{F}$$

$$\vec{S} = S\vec{j}\;;\quad \vec{F} = F\vec{i}$$

$$m\vec{g} = mg(-\sin\alpha\,\vec{i} - \cos\alpha\,\vec{j})$$

$$\left.\begin{aligned} m\ddot{x} &= -mg\sin\alpha + F\\ m\ddot{y} &= -mg\cos\alpha + S\end{aligned}\right]$$

$$\to \quad \frac{1}{2}mr^2\ddot{\varphi} = \vec{N}(P_*)\cdot\vec{k} = rF$$

x, y, F, S, φ

$$\vec{N}(P_*) = \vec{0} + \vec{0} + (-r\vec{j}) \times (F\vec{i}) = rF\vec{k}$$

kotaljenje.

$$\vec{0} = \vec{v}_* + \vec{\omega} \times (P - P_*) = \vec{v}_* + \vec{\omega} \times (-r\vec{j})$$

$$\vec{0} = (\dot{x}\vec{i} + \dot{y}\vec{j}) + \dot{\varphi}\vec{k} \times (-r\vec{j}) = \dot{x}\vec{i} + \dot{y}\vec{j} - r\dot{\varphi}(-\vec{i})$$

$$\left.\begin{aligned} 0 &= \dot{x} + r\dot{\varphi} \\ 0 &= \dot{y} \end{aligned}\right] \qquad \dot{x} = -r\dot{\varphi} \Rightarrow \ddot{\varphi} = -\frac{\ddot{x}}{r}$$

$$S = mg\cos\alpha \qquad -\frac{1}{2}mr^2\frac{\ddot{x}}{r} = rF \Rightarrow F = -\frac{1}{2}m\ddot{x}$$

$$m\ddot{x} = -mg\sin\alpha - \frac{1}{2}m\ddot{x} \qquad \frac{3}{2}\ddot{x} = -g\sin\alpha$$

$$\ddot{x} = -\frac{2}{3}g\sin\alpha \qquad x = -\frac{1}{3}g\sin\alpha\, t^2 \qquad x(t=0) = 0, \quad \dot{x}(t=0) = 0$$

$$F = -\frac{1}{2}m\ddot{x} = \frac{1}{3}mg\sin\alpha \qquad S = mg\cos\alpha$$

$$F \leq kS \Rightarrow \frac{1}{3}mg\sin\alpha \leq kmg\cos\alpha \Rightarrow \boxed{tg\alpha < 3k}$$

b) Kotaljenje z drsenjem

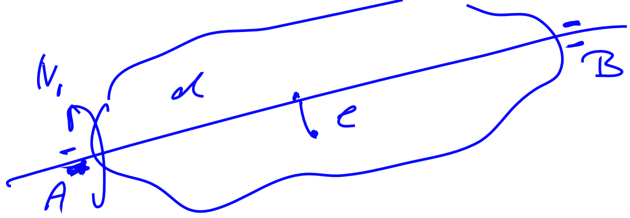
$$\ddot{y} = 0$$

$$F = kS = kmg\cos\alpha$$

$$m\ddot{x} = -mg\sin\alpha + kmg\cos\alpha \Rightarrow \ddot{x} = g(k\cos\alpha - \sin\alpha)$$

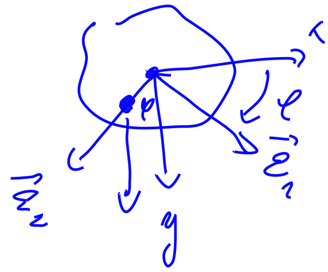
$$\ddot{x} < 0 \qquad k\cos\alpha - \sin\alpha < 0 \Rightarrow k < tg\alpha$$

$$\frac{1}{2}mr^2\ddot{\varphi} = rF = rkmg\cos\alpha \Rightarrow \ddot{\varphi} = \frac{2kg\cos\alpha}{r}$$



$\overline{AB} = \ell$

Vrtenje okrog stalne osi



$$\longrightarrow m\vec{a}_* = m\vec{g} + \vec{A} + \vec{B}$$

$$\vec{e}_r = \vec{\varepsilon}_2 ; \quad \vec{e}_\varphi = -\vec{\varepsilon}_1$$

$$m\vec{g} = mg\vec{j}$$

$$\vec{a}_* = -r\dot{\varphi}^2 \vec{\varepsilon}_2 + r\ddot{\varphi}(-\vec{\varepsilon}_1)$$

$$\vec{j} = \sin\varphi\,\vec{\varepsilon}_1 + \cos\varphi\,\vec{\varepsilon}_2$$

$$\left[\begin{array}{l} -mr\ddot{\varphi} = mg\sin\varphi + A_1 + B_1 \Rightarrow A_1 = A_1(\varphi) \\ -mr\dot{\varphi}^2 = mg\cos\varphi + A_2 + B_2 \Rightarrow A_2 = A_2(\varphi) \\ 0 = A_3 + B_3 \Longrightarrow B_3 = -A_3 \end{array}\right.$$

$$\left|\left|\begin{array}{l} J_{13}\ddot{\varphi} - J_{23}\dot{\varphi}^2 = N_1(A) \quad ; \quad \underline{\underline{J}} = \underline{\underline{J}}(A) \\ J_{23}\ddot{\varphi} + J_{13}\dot{\varphi}^2 = N_2(A) \\ J_{33}\ddot{\varphi} = N_3(A) + N_0 \quad ; \quad N_0 \text{ navor motorja} \end{array}\right.\right.$$

$$\vec{N}(A) = \ell\vec{\varepsilon}_3 \times (B_1\vec{\varepsilon}_1 + B_2\vec{\varepsilon}_2 + B_3\vec{\varepsilon}_3) + (d\vec{\varepsilon}_3 + e\vec{\varepsilon}_2) \times mg(\sin\varphi\,\vec{\varepsilon}_1 + \cos\varphi\,\vec{\varepsilon}_2)$$

$$= \ell B_1\vec{\varepsilon}_2 - \ell B_2\vec{\varepsilon}_1 + mg(d\sin\varphi\,\vec{\varepsilon}_2 - d\cos\varphi\,\vec{\varepsilon}_1 - e\sin\varphi\,\vec{\varepsilon}_3)$$

$$\left[\begin{array}{l} J_{13}\ddot{\varphi} - J_{23}\dot{\varphi}^2 = -\ell B_2 - mgd\cos\varphi \\ J_{23}\ddot{\varphi} + J_{13}\dot{\varphi}^2 = \ell B_1 + mgd\sin\varphi \\ J_{33}\ddot{\varphi} = -mge\sin\varphi + N_0 \Longrightarrow \varphi = \varphi(t) \end{array}\right.$$

$$\left.\begin{array}{l} \\ \\ \end{array}\right\} \Rightarrow B_1 = B_1(t), \quad B_2 = B_2(t)$$

$$\vec{F} = \vec{0}; \quad \vec{N} = \vec{0}$$

$$\vec{N}(A) = \vec{N}(B) + (B-A)\times\vec{F}$$

Prosta vrtavka

Enakomerna rotacija okrog glavnih osi

$$m\vec{a}_* = \vec{F} = \quad \vec{a}_* = \vec{0}$$

$$\boxed{\begin{aligned} J_1\dot{\omega}_1 - \omega_2\omega_3(J_2 - J_3) &= 0 \\ J_2\dot{\omega}_2 - \omega_3\omega_1(J_3 - J_1) &= 0 \checkmark \\ J_3\dot{\omega}_3 - \omega_1\omega_2(J_1 - J_2) &= 0 \checkmark \end{aligned}}$$

$$\underline{\underline{J}} = \underline{\underline{J}}(P_*)$$

$$\omega_1 = \omega_2 = \omega_3 = 0 \checkmark \qquad \omega_2 = \omega_3 = 0; \quad \omega_1 = konst$$

sn, cn, dn Jacobijeve eliptične funkcije

$$(sn(t))^{\cdot} = cn(t)\,dn(t) \qquad sn(t, m)$$

$$T = konst; \qquad \frac{1}{2}\vec{\omega}\cdot\underline{\underline{J}}\vec{\omega} = T_0$$

$$\frac{1}{2}J_1\omega_1^2 + \frac{1}{2}J_2\omega_2^2 + \frac{1}{2}J_3\omega_3^2 = \underline{T_0} \longleftarrow$$

$$\frac{d\vec{\Gamma}}{dt} = \vec{0} = \vec{\Gamma}\ konst; \qquad |\vec{\Gamma}|^2 = \Gamma^2$$

$$\vec{\Gamma} = \underline{\underline{J}}\vec{\omega}; \qquad (J_1\omega_1)^2 + (J_2\omega_2)^2 + (J_3\omega_3)^2 = \underline{\Gamma^2}$$

Binetov elipsoid, presek Binetovega elipsoida s sfero; gibanje vektorja vrtilne količine po Binietovem elipsoidu.

$$\left[\begin{aligned} &\Gamma_1^2 + \Gamma_2^2 + \Gamma_3^2 = \Gamma^2 \\ &\frac{\Gamma_1^2}{2J_1T_0} + \frac{\Gamma_2^2}{2J_2T_0} + \frac{\Gamma_3^2}{2J_3T_0} = 1 \end{aligned}\right. \longleftarrow \text{sfera s polmerom } \Gamma$$

$$a_i = \sqrt{2J_iT_0}$$

$$\frac{\Gamma_1^2}{a_1^2} + \frac{\Gamma_2^2}{a_2^2} + \frac{\Gamma_3^2}{a_3^2} = 1 \qquad \underline{\text{Binetov elipsoid}}$$

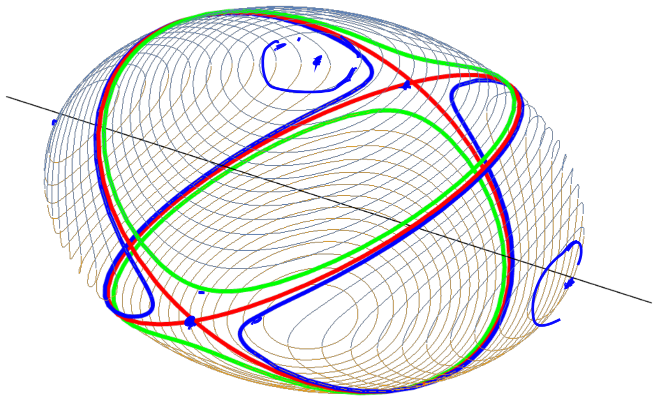
$\vec{\Gamma}$ leži na preseku sfere + Binetovega elipsoida

$\alpha_1 > \alpha_2 > \alpha_3$

$\ell = \alpha_1$

Stabilnost enakomerne rotacije.

Enakomerna rotacija okoli najmanjše glavne smeri je stabilna.



Slika 1: Preske sfere in Binetovega elipsoida.

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Osnosimetrična prosta vrtavka: analitična obravnava; precesija vektorja kotne hitrosti okoli osi.