Predavanje 11. januar 2021

Osnosimetrična prosta vrtavka: analitična obravnava; precesija vektorja kotne hitrosti okoli osi.

$$J_1 \dot{\omega}_1 - \omega_2 \omega_3 (J_2 - J_3) = 0$$

$$J_2 \dot{\omega}_2 - \omega_3 \omega_1 (J_3 - J_1) = 0$$

$$J_3 \dot{\omega}_3 - \omega_1 \omega_2 (J_1 - J_2) = 0 \qquad J_1 = J_2 \neq J_3$$

$$J_3 \dot{\omega}_3 = 0 \Rightarrow \omega_3 = \text{konst}$$

$$\dot{\omega}_1 - \frac{(J_1 - J_3)\omega_3}{J_1} \omega_2 = 0 \qquad \Omega = \frac{J_1 - J_3}{J_1} \omega_3$$

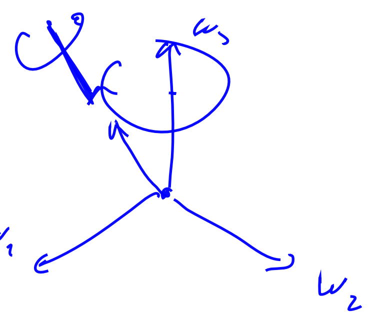
$$\dot{\omega}_2 - \frac{(J_3 - J_1)\omega_3}{J_1} \omega_1 = 0$$

$$\dot{\omega}_1 - \Omega \omega_2 = 0 \qquad \dot{\omega}_2 + \Omega \omega_1 = 0$$

$$\ddot{\omega}_1 = \Omega \dot{\omega}_2 = -\Omega^2 \omega_1 \Rightarrow \ddot{\omega}_1 + \Omega^2 \omega_1 = 0$$

$$\omega_1 = A \cos(\Omega t - \delta) \qquad \omega_2 = \frac{1}{\Omega} \dot{\omega}_1 = -A \sin(\Omega t - \delta)$$

$$\omega_2(t=0) = 0 \Rightarrow \delta = 0 \qquad \omega_1^2 + \omega_2^2 = A^2$$

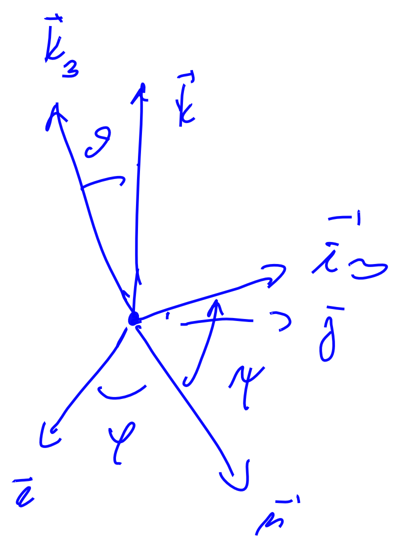


$\vec{\omega}$ precesira okoli osi $\vec{e}_3$ s frekvenco Ω.

$$\vec{v} = \cancel{\vec{v}_t}^{=0} + \vec{\omega} \times \vec{\rho} + \cancel{\vec{v}_{rel}}^{0}$$

$$\vec{\rho} \parallel \vec{\omega} \Rightarrow \vec{v} = 0$$

Eulerjevi koti



$R \neq R(\vec k, \alpha)$

$$\left.\begin{aligned} R\vec i &= \vec i_3 \\ R\vec j &= \vec j_3 \\ R\vec k &= \vec k_3 \end{aligned}\right|$$

$\vec k \times \vec k_3 = \vec n$

φ precesija

ϑ nutacija

ψ kot lastne rotacije

$$R = R(\vec k_2, \psi) \circ R(\vec n, \vartheta) \circ R(\vec k, \varphi)$$

$\vec i_2 \to \vec i_3$	$\vec i_1 \to \vec i_2 = \vec i_1$	$\vec i \mapsto \vec n = \vec i_1$
$\vec j_2 \to \vec j_3$	$\vec j_1 \to \vec j_2$	$\vec j \mapsto \vec j_1$ $\quad \vec k \to \vec k_1 = \vec k$
$\vec k_2 \to \vec k_3 = \vec k_2$	$\vec k_1 \mapsto \vec k_2$	

$$R = \underline{R(\vec k_2, \psi) \circ R(\vec n, \vartheta) \circ R(\vec k, \varphi)} \quad \text{Eulerjeva rotacija}$$

Zapis je enoličen, če $\vartheta \in (0, \pi)$, $\varphi \in [0, 2\pi)$, $\psi \in [0, 2\pi)$,

tj. R ni rotacija okrog osi $\vec k$.

$$R = R(\vec k, \alpha) \Rightarrow \vartheta = 0; \quad \underline{\varphi} + \underline{\psi} = \alpha$$

Velja: $\underbrace{R(\vec k_2, \psi) \circ R(\vec n, \vartheta) \circ R(\vec k, \varphi)} = \underbrace{R(\vec k, \varphi) R(\vec i, \vartheta) \circ R(\vec k, \psi)}$

$$R(\vec k, \psi) = \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Vektor kotne hitrosti

Trditev 1. *Vektor kotne hitrosti Eulerjeve rotacije zapisan v telesni bazni* $(\vec{\epsilon}_1, \vec{\epsilon}_2, \vec{\epsilon}_3)$ *je*

$$\vec{\omega} = (\dot{\varphi}\sin\theta\sin\psi + \dot{\theta}\cos\psi)\vec{\epsilon}_1 + (\dot{\varphi}\sin\theta\cos\psi - \dot{\theta}\sin\psi)\vec{\epsilon}_2 + (\dot{\varphi}\cos\theta + \dot{\psi})\vec{\epsilon}_3.$$

$$N_1 = J_1 \dot{\omega}_1 - \omega_2 \omega_3 (J_2 - J_3) =$$

$$= J_1 \big(\ddot{\varphi} \sin\vartheta \sin\psi + \dot{\varphi}\dot{\vartheta} \cos\vartheta \sin\psi + \dot{\varphi}\dot{\psi} \sin\vartheta \cos\psi + \ddot{\vartheta} \cos\psi - \dot{\vartheta}\dot{\psi} \sin\psi \big) - \dots$$

$\varphi = \varphi(t)$, $\vartheta = \vartheta(t)$, $\psi = \psi(t)$ $\qquad \vartheta \to 0$

Lagrangeeva mehanika

Lagrangeeva funkcija, Lagrangeeve enačbe.

T, U ; $\quad L = T - U \qquad q = (q_1, \ldots, q_n)$

prostostne stopnje

$L = L(q, \dot{q}, t)$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k} = 0 \; ; \quad k = 1, \ldots, n$$

$q = (x, y, z) \qquad T = \frac{1}{2} m v^2 = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$

$\vec{F} = -\operatorname{grad} U$

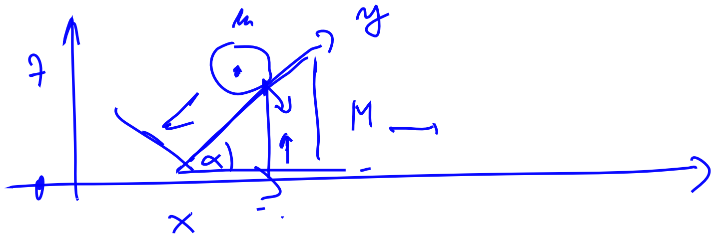
Primer: gibanje proste materialne točke pod vplivom potencialne sile.

$$L = T - U = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - U(x, y, z)$$

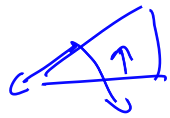
$$\frac{\partial L}{\partial \dot{x}} = m\dot{x} \; ; \quad \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = m\ddot{x} + \frac{\partial U}{\partial x} = 0 \Rightarrow m\ddot{x} = -\frac{\partial U}{\partial x}$$

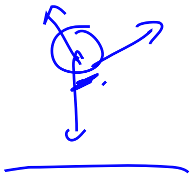
$$m\ddot{y} = -\frac{\partial U}{\partial y} \; ; \quad m\ddot{z} = -\frac{\partial U}{\partial z} \qquad m\vec{a} = -\operatorname{grad} U = \vec{F}$$

Lagrangeeve enačbe so Newtonove enačbe



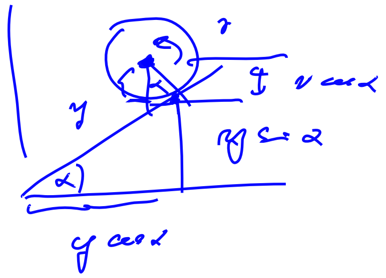
Primer: Kotaljenje valja na drseči strmini.





$$T = \frac{1}{2} M \dot{x}^2 + \frac{1}{2} m v_*^2 + \frac{1}{2} J \dot{\varphi}^2$$

$$U = m g z_*$$



$$z_* = y \sin\alpha + r \cos\alpha$$

$$x + y\cos\alpha - r\sin\alpha$$

$$r\dot{\varphi} = -\dot{y}$$

$$P_*(x + y\cos\alpha - r\sin\alpha,\ z_*)$$

$$v_*^2 = (\dot{x} + \dot{y}\cos\alpha)^2 + (\dot{y}\sin\alpha)^2 = \dot{x}^2 + 2\dot{x}\dot{y}\cos\alpha + \dot{y}^2$$

$$\dot{\varphi} = -\frac{\dot{y}}{r}$$

$$L = \left(\frac{1}{2}(M+m)\dot{x}^2 + m\dot{x}\dot{y}\cos\alpha + \frac{1}{2} m \dot{y}^2 + \frac{1}{2}\,\frac{1}{2} m r^2 \cdot \frac{\dot{y}^2}{r^2}\right)$$

$$- mg(y\sin\alpha + r\cos\alpha)$$

$$L = \frac{1}{2}(M+m)\dot{x}^2 + \frac{3}{4} m \dot{y}^2 + m\dot{x}\dot{y}\cos\alpha - mg\sin\alpha\, y - mgr\cos\alpha$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = \frac{d}{dt}\left((M+m)\dot{x} + m\dot{y}\cos\alpha\right) = (M+m)\ddot{x} + m\cos\alpha\,\ddot{y} = 0$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{y}} - \frac{\partial L}{\partial y} = \frac{d}{dt}\left(\frac{3}{2} m\dot{y} + m\dot{x}\cos\alpha\right) + mg\sin\alpha = 0$$

$$\frac{3}{2} m\ddot{y} + m\ddot{x}\cos\alpha + mg\sin\alpha = 0$$

$$M \to \infty \implies \ddot{x} \to 0 \implies \frac{3}{2} m\ddot{y} = -mg\sin\alpha$$

$$\ddot{x} = -\frac{m\cos\alpha\,\ddot{y}}{M+m}$$

$$\left(\frac{3}{2}\cancel{m} - \frac{m^2\cos^2\alpha}{(m+M)}\right)\ddot{y} = -\cancel{m}g\sin\alpha$$

$$\underbrace{\frac{\frac{3}{2}(m+M) - m\cos^2\alpha}{m+M}}_{\beta}\ddot{y} = -g\sin\alpha$$

$$\boxed{\ddot{y} = -\frac{g}{\beta}\sin\alpha}$$

$L = L(\dot{x}, y, \dot{y})$ $\qquad \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = 0 \qquad \underline{\frac{\partial L}{\partial \dot{x}} = \text{konst}}$

x - ciklična spremenljivka.

$\Downarrow$ $\frac{\partial L}{\partial \dot{x}}$ je konstanta gibanja

$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - U(y, z)$ $\Rightarrow$ $\frac{\partial L}{\partial \dot{x}}$ je konstanta gibanja

$\frac{\partial L}{\partial \dot{x}} = m\dot{x}$ moment

V splošnem primeru $\frac{\partial L}{\partial \dot{q}}$ generalizirani moment.

$L = L(t, q, \dot{q})$