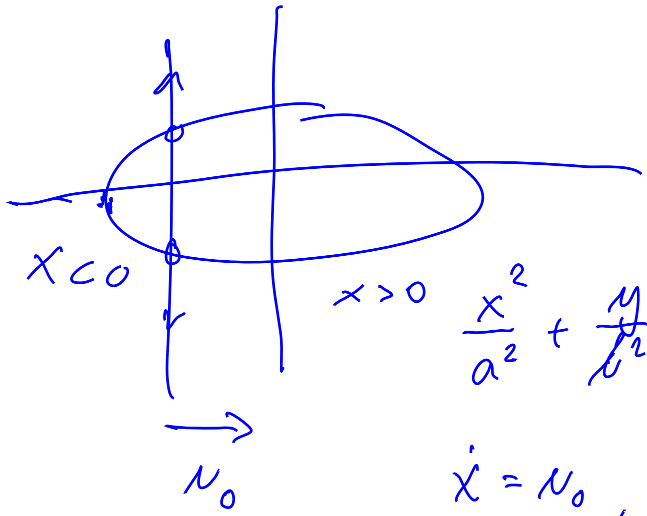
Vaje 13. oktober 2020

1. Vodilo se giblje enakomerno v smeri osi elipse. Določi gibanje presečišča in izračunaj hitrost in pospešek.



$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \quad \text{odvajamo pot}$$

$$\dot{x}=v_0, \quad \ddot{x}=0$$

$$y=\pm b\sqrt{1-\frac{x^2}{a^2}}$$

$$\frac{2x\dot{x}}{a^2}+\frac{2y\dot{y}}{b^2}=0$$

$$\dot{y}=-\frac{b^2}{a^2}\frac{x}{y}\cdot v_0$$

$$y\neq 0$$

$$x=v_0 t+x_0$$

$$x=-a+v_0 t$$

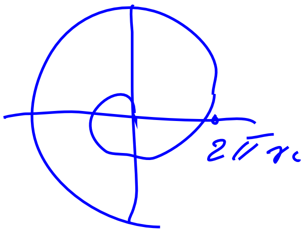
$$y=y(t)$$

$y=0$

$$\frac{2x\dot{x}}{a^2}=0 \qquad x=-a,\ \dot{x}=v_0 \qquad -\frac{2v_0}{a}=0 \quad \times$$

$$\ddot{y}=-\frac{b^2 v_0}{a^2}\,\frac{\dot{x}y-x\dot{y}}{y^2}=\dots$$

2. Točka se giblje po Arhimedovi spirali $r = r_0\varphi$, $\varphi = \omega t$. Izračunaj $\vec{v}$, $\vec{a}$ in dolžino loka spirale od $\varphi = 0$ do $\varphi = 2\pi$.



$$\vec r = r\vec e_r$$

$$\vec v = \dot r \vec e_r + r\vec e_\varphi \dot\varphi =$$

$$\dot r = r_0\dot\varphi = r_0\omega$$

$$= r_0\omega \vec e_r + r_0\varphi\omega\vec e_\varphi =$$

$$= r_0\omega(\vec e_r + \varphi\vec e_\varphi)$$

$$v = |\vec v| = r_0|\omega|\sqrt{1+\varphi^2}$$

$$\vec a = \dot{\vec v} = r_0\omega(\dot\varphi\vec e_\varphi + \dot\varphi\vec e_\varphi + \varphi(-\vec e_r)\dot\varphi) =$$

$$= \underbrace{r_0\omega^2(-\varphi\vec e_r + 2\vec e_\varphi)}$$

$$\dot{\vec e}_\varphi = \frac{d\vec e_\varphi}{d\varphi}\cdot\frac{d\varphi}{dt}; \quad \frac{d\vec e_\varphi}{d\varphi} = -\vec e_r$$

$$[\vec a] = \frac{m}{s^2} \qquad [r_0\omega^2] = m\frac{1}{s^2}$$

$$a = |\vec a| = r_0\omega^2\sqrt{\varphi^2+4}$$

$$l = \int_0^{2\pi}\left|\frac{d\vec r}{d\varphi}\right|d\varphi = \int_0^{2\pi}\left|\frac{d\vec r}{dt}\frac{dt}{d\varphi}\right|d\varphi = \int_0^{2\pi} v\left|\frac{1}{\dot\varphi}\right|d\varphi = \frac{1}{\omega}\int_0^{2\pi} v\,d\varphi$$

$$= \frac{1}{\omega}\int_0^{2\pi} r_0\omega\sqrt{1+\varphi^2}\,d\varphi = r_0\int_0^{2\pi}\sqrt{1+\varphi^2}\,d\varphi$$

$$\int_0^{2\pi}\sqrt{1+\varphi^2}\,d\varphi = \varphi\sqrt{1+\varphi^2}\Big|_0^{2\pi} - \int_0^{2\pi}\frac{\varphi^2+1-1}{\sqrt{1+\varphi^2}}\,d\varphi = 2\pi\sqrt{1+4\pi^2} - \int_0^{2\pi}\sqrt{1+\varphi^2} + \int_0^{2\pi}\frac{d\varphi}{\sqrt{1+\varphi^2}}$$

$$u = \sqrt{1+\varphi^2} \qquad du = \frac{2\varphi}{2\sqrt{1+\varphi^2}}\,d\varphi$$

$$dv = d\varphi \qquad v = \varphi$$

$$\int_0^{2\pi}\sqrt{1+\varphi^2}\,d\varphi = \pi\sqrt{1+4\pi^2} + \frac{1}{2}\int_0^{2\pi}\frac{d\varphi}{\sqrt{1+\varphi^2}} = \pi\sqrt{1+4\pi^2} + \frac{1}{2}\operatorname{arsh}\varphi\Big|_0^{2\pi}$$

$$= \pi\sqrt{1+4\pi^2} + \frac{1}{2}\operatorname{arsh} 2\pi$$

3. Točka se giblje v ravnini tako, da je njena radialna hitrost enaka dvakratni obodni, radialni pospešek pa je enak obodnemu. Določi trajektorijo in tir gibanja.

$$v_r = 2v_\varphi \qquad \dot r = 2r\dot\varphi$$
$$a_r = a_\varphi \qquad \ddot r - r\dot\varphi^2 = r\ddot\varphi + 2\dot r\dot\varphi$$

$$\ddot r = 2\dot r\dot\varphi + 2r\ddot\varphi$$

$$\cancel{2\dot r\dot\varphi} + 2r\ddot\varphi - r\dot\varphi^2 = r\ddot\varphi + \cancel{2\dot r\dot\varphi}$$

$$r\ddot\varphi = r\dot\varphi^2 \qquad \boxed{\ddot\varphi = \dot\varphi^2}$$

$$r = 0 \Rightarrow \dot r = 0 \text{ in } \ddot r = 0 \qquad \dot\varphi = u$$

$$\dot u = u^2 \qquad \int\frac{du}{u^2} = \int dt \Rightarrow -\frac{1}{u} = t + C_1$$

$$\dot\varphi = -\frac{1}{t+C_1} \Rightarrow \int d\varphi = -\int\frac{dt}{t+C_1}$$

$$\varphi = -\log(t+C_1) + \log C_2 = \log\frac{C_2}{t+C_1}$$

$$\varphi(t=0) = 0, \qquad 0 = \log\frac{C_2}{C_1} \Rightarrow C_1 = C_2$$

$$\dot\varphi(t=0) = \omega_0 \qquad \omega_0 = -\frac{1}{C_1} \Rightarrow C_1 = -\frac{1}{\omega_0}$$

$$\varphi = \log\frac{-\frac{1}{\omega_0}}{t-\frac{1}{\omega_0}} = \underline{\log\frac{1}{1-\omega_0 t}}$$

$$\dot r = 2r\dot\varphi = 2r\frac{-1}{t-\frac{1}{\omega_0}} = 2r\frac{\omega_0}{1-t\omega_0}$$

$$\int\frac{dr}{r} = \int 2\frac{\omega_0}{1-t\omega_0}\,dt$$

$$\log\left(r\frac{1}{C_3}\right) = 2\omega_0\log(1-\omega_0 t)\left(-\frac{1}{\omega_0}\right) = -2\log(1-\omega_0 t)$$
$$= \log\frac{1}{(1-\omega_0 t)^2}$$

$$\log(1-\omega_0 t) = -\varphi$$

$$r = \frac{C_3}{(1-\omega_0 t)^2} = \underline{\frac{r_0}{(1-\omega_0 t)^2}} \; ; \qquad \underline{1-\omega_0 t = e^{-\varphi}}$$

$$r(t=0) = r_0$$

$$r = \frac{r_0}{e^{-2\varphi}} = r_0 e^{2\varphi}$$

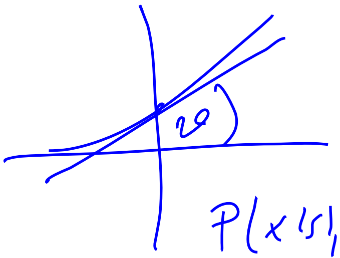
4. Točka se giblje v ravnini tako, da sta obodna hitrost in obodni pospešek konstantna. Določi trajektorijo in tir gibanja.

$v_\varphi = A \quad , \quad a_\varphi = B$

$r\dot{\varphi} = A \quad , \quad r\ddot{\varphi} + 2\dot{r}\dot{\varphi} = B$

$\dot{r}\dot{\varphi} + r\ddot{\varphi} = 0$

5. Pokaži, da za ravninsko krivuljo velja $\kappa = |d\theta/ds|$, kjer je θ kot, ki ga oklepa tangenta z abcisno osjo.



$$\vec{t} = \frac{dx}{ds}\vec{i} + \frac{dy}{ds}\vec{j} = x'\vec{i} + y'\vec{j}$$

$P(x(s), y(s))$

$$\operatorname{tg}\vartheta = \frac{y'}{x'}$$

$$|\vec{t}| = 1$$

$$\vartheta = \operatorname{arctg}\frac{y'}{x'}$$

$$\frac{d\vartheta}{ds} = \frac{1}{1+\left(\frac{y'}{x'}\right)^2}\,\frac{y''x' - y'x''}{x'^2} = \frac{y''x' - y'x''}{x'^2 + y'^2}$$

$$\frac{d\vartheta}{ds} = y''x' - y'x''$$

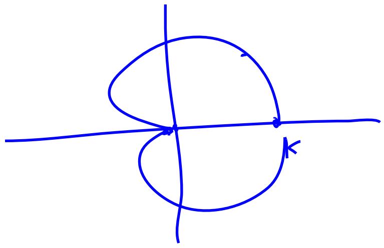
$$\kappa = |\vec{r}\,' \times \vec{r}\,''| \quad ; \quad \kappa = \frac{|\dot{\vec{r}} \times \ddot{\vec{r}}|}{|\dot{s}|^3}$$

$$(x'\vec{i} + y'\vec{j}) \times (x''\vec{i} + y''\vec{j}) = -y'x''\vec{k} + x'y''\vec{k}$$

$$\kappa = |x'y'' - y'x''|$$

6. Točka se giblje po kardoidi $r = k(1+\cos\varphi)$ s konstantno brzino. Izrančunaj:

- kotno hitrost;
- radialni pospešek;
- velikost pospeška.



$|\vec{v}| = v_0$ konst.

$\vec{v} = \dot{r}\,\vec{e}_r + r\dot{\varphi}\,\vec{e}_\varphi \qquad \dot{r}^2 + r^2\dot{\varphi}^2 = v_0^2$

$\dot{r} = \frac{d}{dt}\left(k(1+\cos\varphi)\right) = -k(\sin\varphi)\dot{\varphi}$

$k^2\sin^2\varphi\,\dot{\varphi}^2 + k^2(1+\cos\varphi)^2\dot{\varphi}^2 = v_0^2$

$k^2\dot{\varphi}^2(\sin^2\varphi + 1 + 2\cos\varphi + \cos^2\varphi) = v_0^2$

$2k^2\dot{\varphi}^2\underbrace{(1+\cos\varphi)}_{2\cos^2\frac{\varphi}{2}} = v_0^2$

$4k^2\dot{\varphi}^2\cos^2\frac{\varphi}{2} = v_0^2 \Rightarrow 2k|\dot{\varphi}|\left|\cos\frac{\varphi}{2}\right| = v_0$

$k>0$

$|\dot{\varphi}| = \frac{v_0}{2k\left|\cos\frac{\varphi}{2}\right|}\ ; \quad \dot{\varphi} > 0$

$\varphi \in [0,\pi) \Rightarrow \dot{\varphi} = \frac{v_0}{2k\cos\frac{\varphi}{2}}$

$\varphi \in (\pi, 2\pi) \Rightarrow \dot{\varphi} = -\frac{v_0}{2k\cos\frac{\varphi}{2}}$

$a_r = \ddot{r} - r\dot{\varphi}^2$

$\ddot{r} = (-k\sin\varphi\cdot\dot{\varphi})^{\cdot} = -k(\cos\varphi\,\dot{\varphi}^2 + \sin\varphi\,\ddot{\varphi})$

$\ddot{\varphi} = -\frac{v_0}{2k}\left(-\frac{-\sin\frac{\varphi}{2}}{\cos^2\frac{\varphi}{2}}\cdot\frac{1}{2}\dot{\varphi}\right) = -\frac{v_0}{4k}\frac{\sin\frac{\varphi}{2}}{\cos^2\frac{\varphi}{2}}\left(-\frac{v_0}{2k\cos\frac{\varphi}{2}}\right)$

$= \frac{v_0^2}{8k^2}\frac{\sin\frac{\varphi}{2}}{\cos^3\frac{\varphi}{2}}$

$\ddot{r} = -k\left(\frac{\cos\varphi\, v_0^2}{4k^2\cos^2\frac{\varphi}{2}} + \underbrace{\sin\varphi}_{2\sin\frac{\varphi}{2}\cos\frac{\varphi}{2}}\frac{v_0^2\sin\frac{\varphi}{2}}{8k^2\cos^3\frac{\varphi}{2}}\right) =$

$$= -\frac{1\,\nu_0}{4k\cos^2\frac{\varphi}{2}}\Big)\,\cos\varphi + \sin^2\frac{\varphi}{2}\Big)$$

$$\cos^2\frac{\varphi}{2} - \sin^2\frac{\varphi}{2} + \sin^2\frac{\varphi}{2}$$

$$\ddot{r} = -\frac{\nu_0^2}{4k}$$

7. Izračunaj ukrivljenosti vijačnice κ in τ.