Predavanje 19. oktober 2020

Krivočrtne koordinate

Difeomorfizem med kartezičnimi koordinatami $\mathbf{x} = (x_1, x_2, x_3)$ in krivočrtnimi $\mathbf{q} = (q_1, q_2, q_3)$.

$x = x(q) \qquad q = q(x)$

$\vec r = \sum_{i=1}^{3} x_i \vec e_i \qquad \frac{\partial \vec r}{\partial q_i} = \vec g_i$

$\vec r = (r\cos\varphi\, \vec i + r\sin\varphi\, \vec j) \qquad \frac{\partial \vec r}{\partial r} = \cos\varphi\, \vec i + \sin\varphi\, \vec j = \vec e_r$

Krivočrtni bazni vektorji $\vec g_i = \frac{\partial \vec r}{\partial q_i}$.

$\frac{\partial \vec r}{\partial \varphi} = -r\sin\varphi\, \vec i + r\cos\varphi\, \vec j = r\vec e_\varphi$

Trditev 1. *Vektorji $\vec g_i$, $i = 1, 2, 3$ so linearno neodvisni.*

Primer $\quad \vec g_1 \cdot (\vec g_2 \times \vec g_3) \neq 0$

Cilindrične koordinate

$$\left| \frac{\partial \vec r}{\partial q_1}, \frac{\partial \vec r}{\partial q_2}, \frac{\partial \vec r}{\partial q_3} \right| = \frac{\partial(x_1, x_2, x_3)}{\partial(q_1, q_2, q_3)} \neq 0$$

Jacobijeva det.

bijekcija

$\vec r = r\cos\varphi\, \vec i + r\sin\varphi\, \vec j + z\vec k$

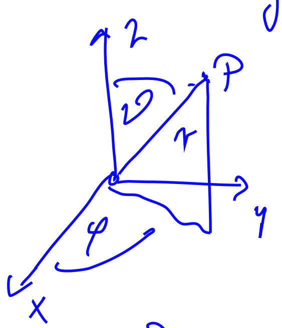
$(r, \varphi, z) \qquad \vec g_r = \frac{\partial \vec r}{\partial r} = \vec e_r$

$\vec g_\varphi = \frac{\partial \vec r}{\partial \varphi} = r\vec e_\varphi \qquad |\vec g_\varphi| = r$

Primer

$\vec g_z = \frac{\partial \vec r}{\partial z} = \vec k \qquad \{\vec g_r, \frac{1}{r}\vec g_\varphi, \vec g_z\}$ ortonormirana

Krogelne koordinate



$x = r\sin\vartheta\cos\varphi \qquad 0 \le r < \infty$

$y = r\sin\vartheta\sin\varphi$

$z = r\cos\vartheta \qquad 0 \le \vartheta \le \pi$

$0 \le \varphi < 2\pi$

$\vec g_r = \frac{\partial \vec r}{\partial r} = \sin\vartheta\cos\varphi\, \vec i + \sin\vartheta\sin\varphi\, \vec j + \cos\vartheta\, \vec k$

$|\vec g_r| = 1$

$\vec g_\vartheta = \frac{\partial \vec r}{\partial \vartheta} = r\cos\vartheta\cos\varphi\, \vec i + r\cos\vartheta\sin\varphi\, \vec j - r\sin\vartheta\, \vec k$

$\left|\frac{\partial \vec r}{\partial \vartheta}\right| = r$

$$\vec g_\varphi = \frac{\partial \vec r}{\partial \varphi} = -r\sin\vartheta\sin\varphi\,\vec i + r\sin\vartheta\cos\varphi\,\vec j$$

$$\left|\frac{\partial \vec r}{\partial \varphi}\right| = r\sin\vartheta$$

$$\frac{\partial \vec r}{\partial r}\cdot\frac{\partial \vec r}{\partial \vartheta} = r\sin\vartheta\cos\vartheta\cos^2\varphi + r\sin\vartheta\cos\vartheta\sin^2\varphi - r\cos\vartheta\sin\vartheta = 0$$

$$\vec g_r\cdot\vec g_\varphi = 0,\quad \vec g_\vartheta\cdot\vec g_\varphi = 0$$

Hitrost v krivočrtnih koordinatah

$$\vec r = x\,\vec i + y\,\vec j + z\,\vec k$$

$$\vec v = \frac{d\vec r}{dt} = \sum_{i=1}^{3}\frac{\partial \vec r}{\partial q_i}\frac{dq_i}{dt} = \frac{\partial \vec r}{\partial q_i}\frac{dq_i}{dt}$$

sumacijska konvencija

V PKS

$$\vec v = \dot r\,\vec g_r + \dot\varphi\,\vec g_\varphi = \dot r\,\vec e_r + \dot\varphi r\,\vec e_\varphi$$

$$\vec v = \sum_{i=1}^{3}\dot q_i\vec g_i.$$

Pospešek v krivočrtnih koordinatah

$$\vec a = \frac{d\vec v}{dt} = \ddot q_i\,\vec g_i + \dot q_i\frac{\partial \vec g_i}{\partial q_j}\dot q_j$$

$$\frac{\partial \vec g_i}{\partial q_j} = \sum_{k=1}^{3}\Gamma^k_{ij}\,\vec g_k = \Gamma^k_{ij}\,\vec g_k$$

Christoffelovi simboli

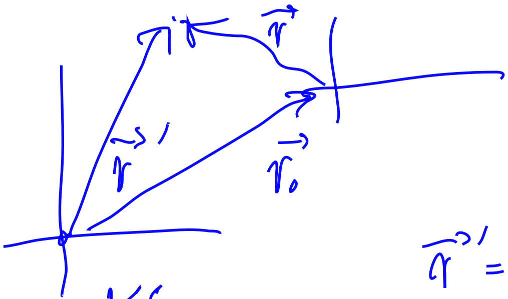
$$\vec a = \ddot q_i\,\vec g_i + \dot q_i\dot q_j\,\Gamma^k_{ij}\,\vec g_k = \left(\ddot q_k + \dot q_i\dot q_j\,\Gamma^k_{ij}\right)\vec g_k$$

$$\vec a = (\ddot r - r\dot\varphi^2)\,\vec e_r + (r\ddot\varphi + 2\dot r\dot\varphi)\,\vec e_\varphi$$

Dinamika točke

Newtonovi zakoni

1) Prvi Newtonov zakon: koordinatni sistem je inercialen natanko tedaj, ko se prosta materialna točka giblje premočrtno s konstantno brzino.



KS

IKS

$$\vec r = \vec c\,t + r(t=0)$$

$$\vec r\,' = \vec r_0 + \vec r$$

$$\vec v\,' = \dot{\vec r}\,' = \dot{\vec r}_0 + \dot{\vec r} = \dot{\vec r}_0 + \vec c$$

$\dot{\vec r}_0 = \vec c\,'$ potem se točka giblje premočrtno tudi v KS.

KS je inercialen

2) Drugi Newtonov zakon: V IKS velja Newtonova enačba

$$m\vec{a} = \vec{F}.$$

$$\vec{F} = \vec{0} \Rightarrow \vec{a} = \vec{0}.$$

3) Tretji Newtonov zakon: $\vec{r}_1, \vec{r}_2$

$$m_1 \vec{a}_1 = \vec{F}_{21}$$

$$m_2 \vec{a}_2 = \vec{F}_{12}$$

$$\vec{F}_{21} = -\vec{F}_{12}$$

$\vec{a} = \ddot{\vec{r}}$; $\vec{F}$ poznamo

$\vec{r}(t = t_0)$, $\dot{\vec{r}}(t = t_0)$ začetna pogoja

Sila + začetna pogoja natanko določata gibanje.

Vprašanje obstoja IKS.

LKS

geocentrični KS GKS

heliocentrični HKS

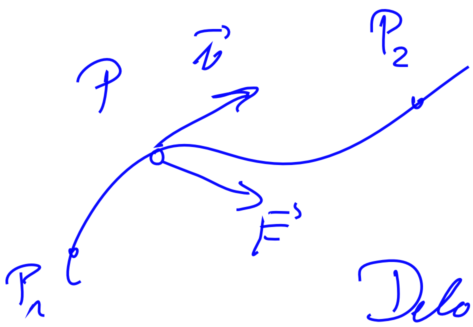
Gibanje je natanko določeno z začetnim pogojem in Newtonovo enačbo.

Delo, ki ga opravi sila pri gibanju od P_1 do P_2

$$\frac{1}{2} m v^2 = T$$

Izrek o delu in energiji

Kinetična energija, delo.



$$A = \int_{P_1}^{P_2} \vec F \cdot d\vec P = \int_{r_1}^{r_2} \vec F \cdot d\vec r = \int_{t_1}^{t_2} \underbrace{\vec F(\vec r(t)) \cdot \vec v}_{\text{moč}}\, dt$$

$$\vec r_1 = \vec r(t = t_1) \qquad d\vec r = \frac{d\vec r}{dt} dt$$

Delo je integral moči.

Izrek 1. Izrek o delu. *Delo, ki ga opravi sila na materialno točo pri gibanju vzdolž tira, je enako razliki kinetične energije.*

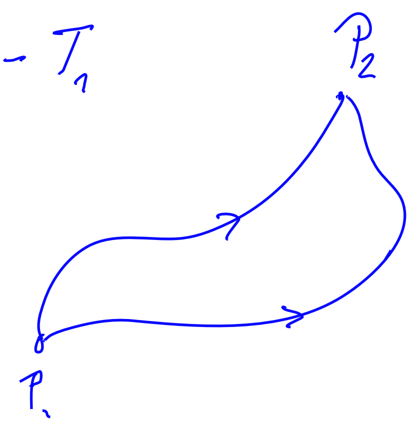
$$A = \int_{t_1}^{t_2} \vec F \cdot \vec v\, dt = \int_{t_1}^{t_2} m \vec a \cdot \vec v\, dt = \frac{1}{2} m \int_{t_1}^{t_2} \frac{d}{dt}(v^2)\, dt =$$

$$\vec a = \frac{d\vec v}{dt} = \dot{\vec v}$$

$$\dot{\vec v} \cdot \vec v = \frac{1}{2} \frac{d}{dt} (\vec v \cdot \vec v)$$

$$= \frac{1}{2} m v_2^2 - \frac{1}{2} m v_1^2 = T_2 - T_1$$

Izrek o delu velja vedno.



$$\vec F = -\operatorname{grad} U$$

obstaja V, da je

Potencialna sila, konzervativna sila.

Sila je konzervativna, če je potencialna in je odvisna samo od položaja

V splošnem $\vec{F} = \vec{F}(t, \vec{r}, \dot{\vec{r}})$ | $V = V(\vec{r})$

Izrek 2. Izrek o energiji. *Naj bo $\vec{F}$ konzervativna sila. Potem je vsota kinetične in potencialne energije konstanta gibanja.*

Osnovni primeri sil

- Gibanje v polju konstantne sile $\vec{F} = \vec{F}_0 = m\vec{g}$.

- Gravitacijska sila $\vec{F} = -\frac{\kappa m_1 m_2}{r^2}\,\vec{r}/r$, potencial je

$$U = -\kappa m_1 m_2/r.$$

- Električna sila.

- Sila vzmeti, Hookova sila.

$$\vec{F} = -k\frac{\vec{r}-\vec{r}_0}{|\vec{r}-\vec{r}_0|}\left(|\vec{r}-\vec{r}_0| - l\right)$$

  Potencial je

$$U = \frac{1}{2}k\left(|\vec{r}-\vec{r}_0| - l\right)^2.$$

- Sila upora, sila trenja.