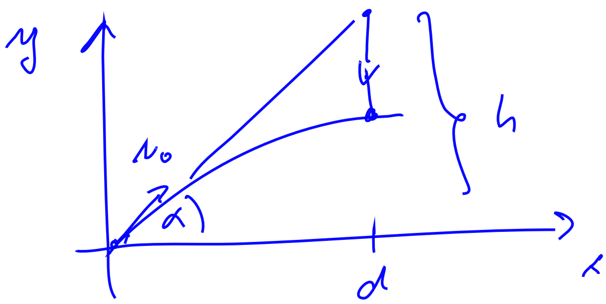
Vaje 27. oktober 2020

1. Določi kot poševnega meta tako, da izstrelek zadane tarčo, ki jo spustimo v prosti pad v trenutku sprožitve izstrelka.



$$x = v_0 \cos\alpha \, t \qquad\qquad x_1 = d$$

$$y = -\frac{1}{2} g t^2 + v_0 \sin\alpha \, t \qquad\qquad y_1 = -\frac{1}{2} g t^2 + h$$

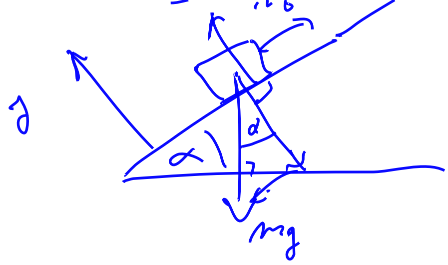
Tarčo zadenemo v $t = t_1$

$$x(t_1) = x_1 \qquad y(t_1) = y_1$$

$$v_0 \cos\alpha \, t_1 = d \Rightarrow t_1 = \frac{d}{v_0 \cos\alpha}$$

$$\cancel{-\frac{1}{2} g t_1^2} + v_0 \sin\alpha \, t_1 = \cancel{-\frac{1}{2} g t_1^2} + h$$

$$\cancel{v_0} \sin\alpha \frac{d}{\cancel{v_0} \cos\alpha} = h \Rightarrow \underline{\operatorname{tg}\alpha = \frac{h}{d}}$$



$$\int_{P_1}^{P_2} \vec F\cdot d\vec r = \int_{t_1}^{t_2} \vec F\cdot\vec v\,dt$$

$\vec S \perp \vec v$

delo sile podlage je enako nič

2. Drsenje klade na strmini. Klado zaženemo navzgor z začetno brzino v_0. Kdaj se ustavi, če je gibanje

   - brez trenja;
   - s trenjem.

$$m\vec g = -mg\sin\alpha\,\vec i - mg\cos\alpha\,\vec j$$

$$m\ddot x = -mg\sin\alpha$$

$U = mg\sin\alpha\, x$ ✓ $\qquad \frac12 m\dot x^2 + U = E_0$

$-U' = -mg\sin\alpha$

$t=0 \Rightarrow x=0$

$$\frac12 m v_0^2 + 0 = E_0$$

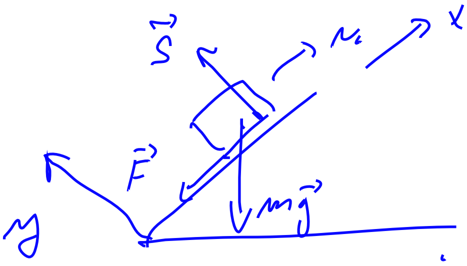
$$\frac12 m\dot x^2 + mg\sin\alpha\, x = \frac12 m v_0^2$$

V najvišji točki je $\dot x = 0 \Rightarrow \underline{x = \frac{v_0^2}{2g\sin\alpha}}$

$x = -\frac12 g\sin\alpha\, t^2 + v_0 t \qquad \dot x(t=0) = v_0$

$$\frac{v_0^2}{2g\sin\alpha} = -\frac12 g\sin\alpha\, t^2 + v_0 t \Rightarrow t^2 - \frac{2v_0}{g\sin\alpha}t + \frac{v_0^2}{g^2\sin^2\alpha} = 0$$

$$t_{1,2} = \frac12\left(\frac{2v_0}{g\sin\alpha} \pm \sqrt{\frac{4v_0^2}{g^2\sin^2\alpha} - \frac{4v_0^2}{g^2\sin^2\alpha}}\right) = \frac{v_0}{g\sin\alpha}$$



$$m\vec a = m\vec g + \vec S + \vec F$$

$$\vec S = S\vec j, \qquad \vec F = -F\vec i$$

$$m\ddot x = -mg\sin\alpha \pm F$$

$$0 = m\ddot y = -mg\cos\alpha + S \Rightarrow S = mg\cos\alpha$$

$\underline{F = kS = kmg\cos\alpha} \qquad m\ddot x = -mg\sin\alpha - kmg\cos\alpha$

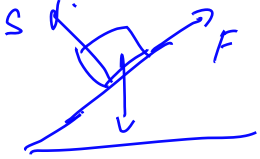
$\ddot x = -g(\sin\alpha \mp k\cos\alpha) \leftarrow \qquad \ddot x < 0 \qquad \sin\alpha - k\cos\alpha > 0$

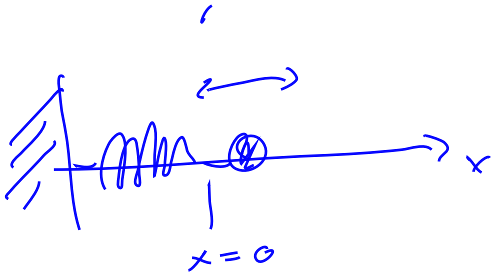
$\dot x = -g(\sin\alpha \mp k\cos\alpha)\,t + v_0 \qquad \tan\alpha > k$

Pri tem pogoju je možno gibanje navzdol

$$x = -\frac12 g(\sin\alpha + k\cos\alpha)t^2 + v_0 t$$

$\dot x = 0 \Rightarrow t_1 = \frac{v_0}{g(\sin\alpha + k\cos\alpha)} \qquad \Big|\ k = 0$ gibanje brez trenja





3. Harmonični oscilator s trenjem.

$$m\ddot{x} = -kx \underbrace{- \alpha \frac{\dot{x}}{|\dot{x}|}}_{\text{sila trenja}} \qquad \alpha > 0$$

(1) $\dot{x} > 0 \qquad t > 0$

$x(t=0) = -a \; ; \quad a > 0$

$\dot{x}(t=0) = 0$

$$m\ddot{x} = -kx - \alpha$$

$$\ddot{x} + \frac{k}{m}x = -\frac{\alpha}{m} \qquad \frac{k}{m} = \omega^2 \; ; \quad \frac{\alpha}{m} = \beta$$

$$\ddot{x} + \omega^2 x = -\beta$$

$$\ddot{x}_0 + \omega^2 x_0 = 0 \Rightarrow x_0 = A\cos\omega t + B\sin\omega t$$

$$x_1 = -\frac{\beta}{\omega^2}$$

$$x = A\cos\omega t + B\sin\omega t - \beta/\omega^2$$

$$\dot{x}(t=0) = 0 \Rightarrow B = 0 \Rightarrow x = A\cos\omega t - \beta/\omega^2$$

$$-a = A - \beta/\omega^2 \Rightarrow A = -a + \beta/\omega^2$$

$$x = (-a + \beta/\omega^2)\cos\omega t - \beta/\omega^2$$

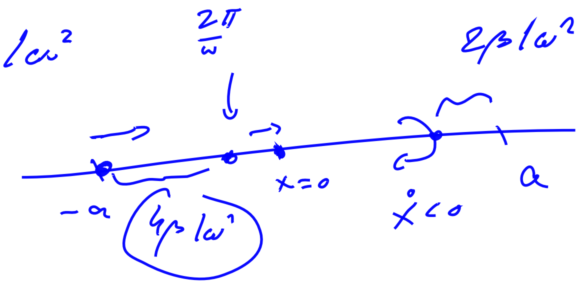
Kdaj in kje se točka ustavi?

$$\dot{x} = -\omega(-a + \beta/\omega^2)\sin\omega t \Rightarrow \dot{x} = 0 \Leftrightarrow \omega t = k\pi$$

$$t_1 = \frac{\pi}{\omega}$$

$$x_1 = x(t_1) = (-a + \beta/\omega^2)(-1) - \beta/\omega^2$$

$$x_1 = a - 2\beta/\omega^2$$

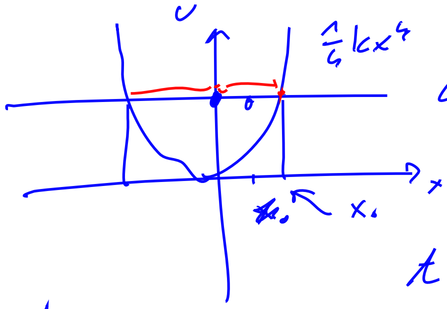


$-kx^3$

4. Materialna točka se giblje po osi x, na točko pa deluje sila $f = -\cancel{k/}x^3$, $k > 0$.

- Ddoloči potencial in kvalitativno obravnavaj gibanje.
- Kdaj pride iz začetnega položaja $x(t=0) = x_0$ v center sile.

$$f = -kx^3 \quad ; \quad f = -\frac{dV}{dx} \Rightarrow \frac{dV}{dx} = kx^3 \Rightarrow V = \frac{1}{4}kx^4$$



E_0 Vsa gibanja so periodična

$$t = -\int_{x_0}^{0} \frac{dx}{\sqrt{\frac{2}{m}(E_0 - V(x))}} = \sqrt{\frac{m}{2}} \int_0^{x_0} \frac{dx}{\sqrt{E_0 - \frac{1}{4}kx^4}}$$

$$= \sqrt{\frac{m}{2E_0}} \int_0^{x_0} \frac{dx}{\sqrt{1 - \frac{k}{4E_0}x^4}} \qquad V(x_0) = E_0 = \frac{1}{4}kx_0^4$$

$$t = \sqrt{\frac{m \cdot 4}{2kx_0^4}} \int_0^{x_0} \frac{dx}{\sqrt{1 - \left(\frac{x}{x_0}\right)^4}} = \sqrt{\frac{2m}{k}} \frac{1}{x_0^2} \int_0^1 \frac{x_0\, dz}{\sqrt{1-z^4}} =$$

$$x = x_0 \cdot z$$

$$t = \sqrt{\frac{2m}{k}} \frac{1}{x_0} \int_0^1 \frac{dz}{\sqrt{1-z^4}} \qquad T = 4t$$

$$\underline{I} = \int_0^1 \frac{dz}{\sqrt{1-z^4}} = \frac{1}{4}\int_0^1 u^{-3/4}(1-u)^{-1/2}\, du$$

$$u = z^4$$
$$du = 4z^3 dz \Rightarrow dz = \frac{1}{4} z^{-3} du = \frac{1}{4} u^{-3/4} du$$
$$z = u^{1/4}$$

$$B(x,y) = \int_0^1 u^{x-1}(1-u)^{y-1}\, du \qquad B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

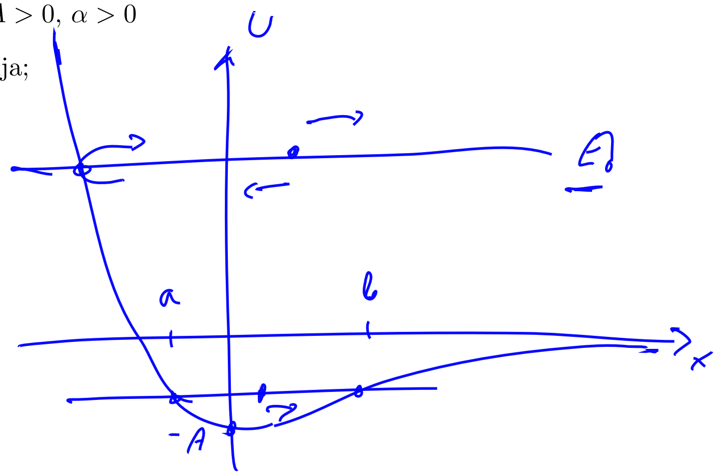
$$\underline{I} = \frac{1}{4} B\left(\frac{1}{4}, \frac{1}{2}\right) = \frac{1}{4} \frac{\Gamma(\frac{1}{4})\Gamma(\frac{1}{2})}{\Gamma(\frac{3}{4})}$$

$$\Gamma(z+1) = z\Gamma(z)$$
$$\Gamma\left(\frac{5}{4}\right) = \frac{1}{4}\Gamma\left(\frac{1}{4}\right)$$

$$t = \sqrt{\frac{2m}{k}} \frac{1}{4x_0} \frac{\Gamma(\frac{1}{4})\Gamma(\frac{1}{2})}{\Gamma(\frac{3}{4})}.$$

5. Za potencial $U(x) = A\left(e^{-2\alpha x} - 2e^{-\alpha x}\right)$, $A > 0$, $\alpha > 0$

- kvalitativno obravnavaj možna gibanja;
- Izračunaj periodo gibanja.



$U(0) = -A$

$U' = A(-2\alpha e^{-2\alpha x} + 2\alpha e^{-\alpha x})$

$e^{-\alpha x} = u$

$0 = -u^2 + u = u(1-u)$

$u=0,\ u=1$

$u = 1 \Rightarrow x = 0$

$E_0 \geq 0$ točka uhaja v $+\infty$

$-A < E_0 < 0$ gibanje je periodično

$E_0 = -A$ točka mirovanja.

$E_0 = U(x_0) \Rightarrow$ v x_0 je $\dot{x} = 0$

V $x=0$ je $\ddot{x} = 0$ $\quad \vec{F} = -\frac{dU}{dx} = -U'$

$U'(x=0) = 0 \Rightarrow \vec{F} = 0 \Rightarrow \ddot{x} = 0$

T (perioda)

$$t = \pm \int_{x_0}^{x} \frac{dx}{\sqrt{\frac{2}{m}(E_0 - U(x))}}$$

$$T = +\int_{x_0}^{b} \frac{dx}{\sqrt{\ }} - \int_{b}^{a} \frac{dx}{\sqrt{\ }} + \int_{a}^{x_0} \frac{dx}{\sqrt{\ }} = 2\int_{a}^{b} \frac{dx}{\sqrt{\ }}$$

$$T = \sqrt{2m} \int_a^b \frac{dx}{\sqrt{E_0 - U(x)}}$$

$$T = \sqrt{2m} \int_{a(E_0)}^{b(E_0)} \frac{dx}{\sqrt{E_0 - A(e^{-2\alpha x} - 2e^{-\alpha x})}} = \sqrt{2m} \int_a^b \frac{dx}{e^{-\alpha x}\sqrt{E_0 e^{2\alpha x} - A + 2Ae^{\alpha x}}}$$

$e^{-2\alpha x}$

$u = e^{\alpha x} \quad du = \alpha e^{\alpha x} dx$ Se meje na krajših

$$T = \sqrt{2m} \int_{\hat{a}}^{\hat{b}} \frac{1}{\alpha} \frac{du}{\sqrt{E_0 u^2 - A + 2Au}} = \sqrt{\frac{2m}{-E_0}} \frac{1}{\alpha} \pi \qquad \frac{dx}{e^{-\alpha x}} = e^{\alpha x} dx$$

kvadraten polinom z vodilnim členom $E_0 < 0$