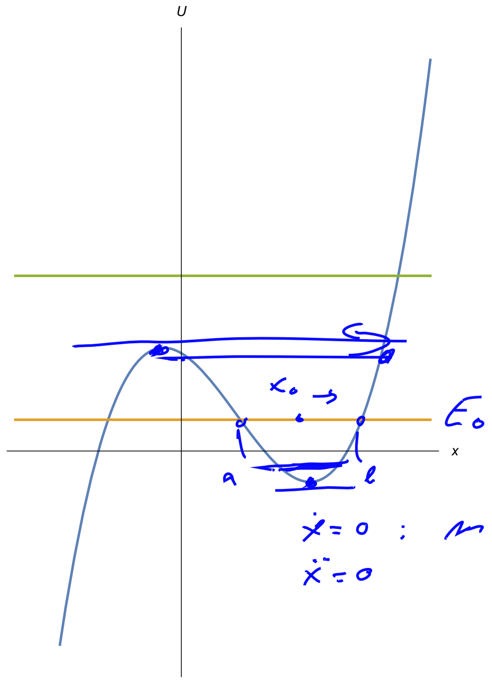
Predavanje 2. november 2020

Kvalitativna analiza gibanja

Ravnovesna lega



$\dot x = 0$: $m\ddot x = -\frac{dU}{dx} = 0$

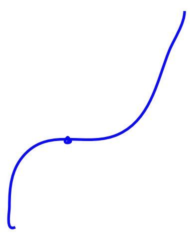
$\ddot x = 0$

Slika 1: Graf potenciala.

Stacionarne točke potenciala so točke mirovanja.

Lokalni minimum je stabilna ravnovesna lega.

maksimum nestabilna



$$t - t_0 = \mathrm{sgn}\,\dot x \int_{x_0}^{x} \frac{dx}{\sqrt{\frac{2}{m}(E_0 - U(x))}}$$

$t_0 = 0$;

t_1 od x_0 do b ; $\dot x > 0$

t_2 od b do a ; $\dot x < 0$

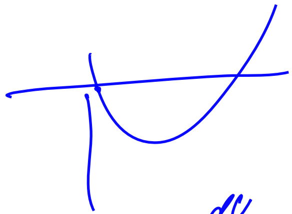
t_3 od a do x_0 ; $\dot x > 0$

$$T = t_1 + t_2 + t_3 = \int_{x_0}^{b} \frac{dx}{\sqrt{\;}} - \int_{b}^{a} \frac{dx}{\sqrt{\;}} + \int_{a}^{x_0} \frac{dx}{\sqrt{\;}} = 2\int_{a}^{b} \frac{dx}{\sqrt{\frac{2}{m}(E_0 - U(x))}}$$

$$T = \sqrt{2m} \int_a^b \frac{dx}{\sqrt{E_0 - U(x)}}$$

Perioda gibanja

$$U(a) = U(b) = E_0$$



$$U(x) = U(a) + \frac{dU}{dx}(a)(x-a) + o(x-a)$$

$$\sqrt{E_0 - U(x)} \doteq \sqrt{-\frac{dU}{dx}(a)(x-a)} = \sqrt{-\frac{dU}{dx}(a)}\sqrt{x-a} \qquad x=a \qquad \frac{dU}{dx}(a)<0$$

$$\frac{dU}{dx}(b)>0$$

$$T = \sqrt{2m} \int_a^b \frac{dx}{\sqrt{E - U(x)}}.$$

$$\int_0^1 \frac{dx}{x^p} \qquad p<1 \qquad \int_a^{a+\delta} \frac{dz}{\sqrt{z-a}} \quad \text{obstaja}$$

Integral obstaja

Lema 1.

$$\boxed{\int_a^b \frac{du}{\sqrt{(b-u)(u-a)}} = \pi.}$$

Primer: harmonični oscilator

V okolici $x=a$ $\qquad \frac{dU}{dx}(a)=0$

$$U(x) \doteq U(a) + \frac{1}{2}U''(a)(x-a)^2$$

$$\sqrt{E_0 - U(x)} \doteq \sqrt{\frac{1}{2}U''(a)}\,(x-a) \quad ; \quad p=1$$

$T=\infty$

$$T = \sqrt{2m} \int_{a(E_0)}^{b(E_0)} \frac{dx}{\sqrt{E_0 - U(x)}}$$

Dokaz leme

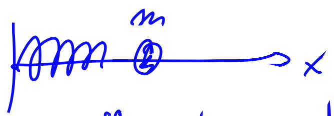
$$u = \frac{1}{2}(a+b) + \frac{1}{2}(b-a)z$$

$$I = \int_a^b \frac{du}{\sqrt{(b-u)(u-a)}} = \int_{-1}^{1} \frac{\frac{1}{2}(b-a)\,dz}{\sqrt{\left(\frac{1}{2}(b-a)\right)^2(1-z^2)}} = \int_{-1}^{1}\frac{dz}{\sqrt{1-z^2}} = \arcsin z\Big|_{-1}^{1} = \pi$$

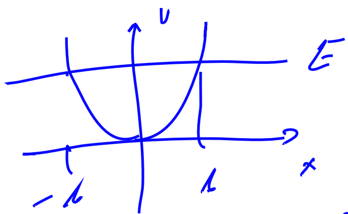
Harmonična aproksimacija

$$b-u = b - \frac{1}{2}(a+b) - \frac{1}{2}(b-a)z =$$
$$= \frac{1}{2}(b-a)(1-z)$$
$$u-a = \frac{1}{2}(b-a)(1+z)$$

Primer HO



$$m\ddot{x} = -kx \quad ; \quad V = \frac{1}{2}kx^2$$



Vsa gibanja so periodična

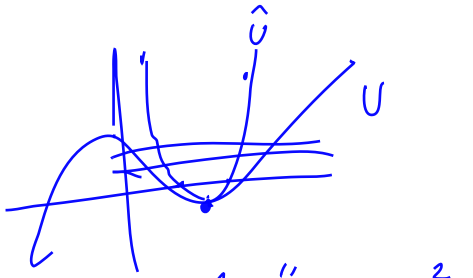
$$T \approx 2\pi\sqrt{m/U''(x_0)}$$

$\ddot{x} + \frac{k}{m}x = 0$; $\omega^2 = \frac{k}{m}$ $\quad x = A\cos \ldots B\sin\omega t$ $\quad T = 2\pi\frac{1}{\omega} = 2\pi\sqrt{\frac{m}{k}}$

$$T = \sqrt{2m}\int_a^b \frac{dx}{\sqrt{E_0 - \frac12 kx^2}} = \sqrt{2m}\int_{-b}^{b}\frac{dx}{\sqrt{\frac12 k}\sqrt{\frac{2E_0}{k}-x^2}} = \sqrt{2m}\sqrt{\frac{2}{k}}\cdot\pi = 2\pi\sqrt{\frac{m}{k}}$$

Primer: Duffingov oscilator

$$U = \frac12\alpha x^2 + \frac14\beta x^4, \qquad \alpha\beta < 0.$$



Harmonična aproksimacija

$$U(x) \doteq U(x_0) + U'(x_0)(x-x_0) + \frac12 U''(x_0)(x-x_0)^2 = \underbrace{U(x_0)+\frac12 U''(x_0)(x-x_0)^2}_{\hat U}$$

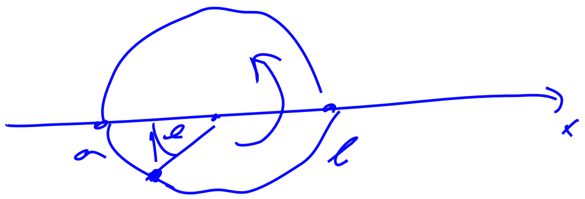
$$T = \sqrt{2m}\int_a^b \frac{dx}{\sqrt{E_0-U(x)}} \doteq \sqrt{2m}\int_{\hat a}^{\hat b}\frac{dx}{\underbrace{\sqrt{E_0 - U(x_0) - \frac12 U''(x_0)(x-x_0)^2}}}$$

v lok. min.

$$= \sqrt{2m}\int_{\hat a}^{\hat b}\frac{dx}{\sqrt{\frac12 U''(x_0)}\sqrt{\alpha+\beta x - x^2}} = \boxed{2\pi\sqrt{\frac{m}{U''(x_0)}}} \qquad U''(x_0)>0$$

Libracijska aproksimacija

Aproksimacija je neodvisna od E_0.



$$t - t_0 = \operatorname{sgn}\dot x \int_{x_0}^{x}\frac{dx}{\sqrt{\frac{2}{m}(E_0-U)}}$$

$E_0 - U = (b-x)(x-a)\chi(x)$ $\qquad \chi(x)>0 \qquad x\in[a,b]$

$x = \frac12(a+b) - \frac12(b-a)\cos\vartheta \qquad dx = \frac12(b-a)\sin\vartheta\, d\vartheta$

$b - x = b - \frac12(a+b) + \frac12(b-a)\cos\vartheta = \frac12(b-a)(1+\cos\vartheta)$

$x - a = \frac12(a+b) - \frac12(b-a)\cos\vartheta - a = \frac12(b-a)(1-\cos\vartheta)$

$E_0 - U = \left(\frac12(b-a)\right)^2(1-\cos^2\vartheta)\chi(\vartheta) \qquad t_0=0,\ x_0=a$

$$t = \sqrt{\frac{m}{2}}\operatorname{sgn}\dot x \int_0^\vartheta \frac{\sin\vartheta\,d\vartheta}{\sqrt{\sin^2\vartheta\,\chi(\vartheta)}} = \operatorname{sgn}\dot x\int_0^\vartheta\frac{\sin\vartheta\,d\vartheta}{|\sin\vartheta|\sqrt{\chi(\vartheta)}}$$

$$T = \sqrt{2m}\int_0^\pi \frac{1}{\sqrt{\chi(\phi)}}\,d\phi \approx \pi\sqrt{\frac{m}{2}}\left(\frac{1}{\sqrt{\chi(a)}}+\frac{1}{\sqrt{\chi(b)}}\right)$$

$\vartheta\in(0,\pi) \qquad \dot x>0$; $\frac{\sin\vartheta}{|\sin\vartheta|}=1 \qquad t = \sqrt{\frac{m}{2}}\int_0^\vartheta \frac{d\vartheta}{\sqrt{\chi(\vartheta)}}$

$\vartheta\in(\pi,2\pi)$; $\dot x<0$; $\sin\vartheta<0 \Rightarrow |\sin\vartheta| = -\sin\vartheta$

$\operatorname{sgn}\dot x = -1$; $\frac{\sin\vartheta}{|\sin\vartheta|} = -$ $\qquad t = \int_0^\pi\frac{d\vartheta}{\sqrt{\chi(\vartheta)}} + \int_\pi^\vartheta \frac{d\vartheta}{\sqrt{\chi(\vartheta)}}$

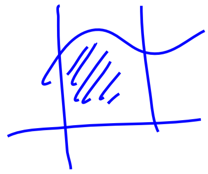
$$\boxed{t = \sqrt{\frac{m}{2}} \int_0^{\vartheta} \frac{d\vartheta}{\sqrt{\chi(\vartheta)}}}$$

$$T = 2\sqrt{\frac{m}{2}} \int_0^{\pi} \frac{d\vartheta}{\sqrt{\chi(\vartheta)}} \qquad = \int_0^{\vartheta} \frac{d\vartheta}{\sqrt{\chi(\vartheta)}}$$

Primer: Duffingov oscilator

Libracijska aproksimacija

$$\int_0^{\pi} \frac{d\vartheta}{\sqrt{\chi(\vartheta)}} \doteq \pi \cdot \frac{1}{2}\left(\frac{1}{\sqrt{\chi(\vartheta=0)}} + \frac{1}{\sqrt{\chi(\vartheta=\pi)}} \right)$$

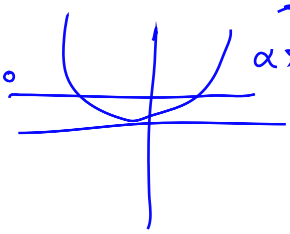


Trapezna formula

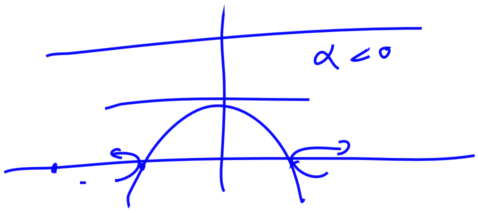
$$\boxed{T \doteq \sqrt{\frac{m}{2}}\, \pi \left(\frac{1}{\sqrt{\chi(a)}} + \frac{1}{\sqrt{\chi(b)}} \right)}$$

$$U = \frac{1}{2}\alpha x^2 + \frac{1}{4}\beta x^4$$

$\alpha\beta > 0$



$\alpha > 0$

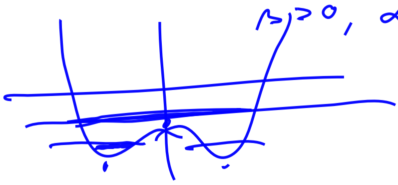


$\alpha < 0$

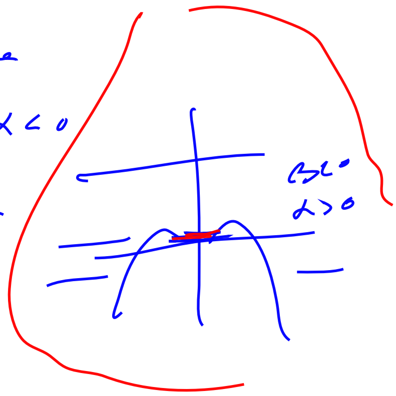
bez gibanja nema gibanja

$\beta > 0, \ \alpha < 0$

$\alpha\beta < 0$



~~Fazni portret~~



$\beta < 0$

$\alpha > 0$

$\beta > 0, \ \alpha < 0$

Harmonična aproksimacija $\quad T = 2\pi \sqrt{\frac{m}{U''(x_0)}}$

$$U' = \alpha x + \beta x^3 = x(\alpha + \beta x^2) \qquad x_0 = 0, \quad x_{1,2} = \pm\sqrt{-\frac{\alpha}{\beta}}$$

$$U'' = \alpha + 3\beta x^2 \qquad x_0 = 0 \qquad U''(0) = \alpha < 0 \qquad T_0 = ?$$

$$U''(x_{1,2}) = \alpha + 3\beta\left(-\frac{\alpha}{\beta}\right) = -2\alpha \qquad \underline{\underline{T_{1,2} = 2\pi\sqrt{\frac{m}{-2\alpha}}}}$$

Primer: ~~harmonični oscilator~~

$$T = \sqrt{\frac{m}{2}} \left(\frac{1}{\sqrt{\chi(a)}} + \frac{1}{\sqrt{\chi(b)}} \right) \pi = \pi\sqrt{2m}\, \frac{1}{\sqrt{\chi(a)}}$$

$T_0 \doteq$

$$\frac{1}{2}\alpha x^2 + \frac{1}{4}\beta x^4 = E_0 > 0 \qquad a = -b \qquad \chi(a) = \chi(b)$$

$$E_0 - U = (b-x)(x-a)\chi(x) \qquad \chi(x) = \frac{E_0 - U}{(b-x)(x-a)} = \frac{E_0 - U}{a^2 - x^2}$$

$x = a$

$$\chi(a) = \lim_{x\to a} \frac{E_0 - U}{a^2 - x^2} = \lim_{x \to a} \frac{-x(\alpha + \beta x^2)}{-2x} = \alpha + \beta a^2 = \alpha - \alpha + \sqrt{\alpha^2 + 4\beta E_0}$$

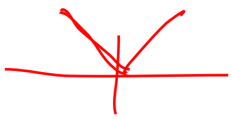
$$\frac{1}{4}\beta x^4 + \frac{1}{2}\alpha x^2 - E_0 = 0 \qquad x^2 = u \qquad u_{1,2} = \frac{-\frac{1}{2}\alpha \pm \sqrt{\frac{1}{4}\alpha^2 + \beta E_0}}{\frac{1}{2}\beta}$$

$$x^2_{1,2} = \frac{-\alpha \pm \sqrt{\alpha^2 + 4\beta E_0}}{\beta} \qquad x^2 = \frac{-\alpha + \sqrt{\alpha^2 + 4\beta E_0}}{\beta}$$

Trditev 1. *Naj bo S ploščina faznega portreta periodičnega gibanja. Potem* $\frac{\mathrm{d}S}{\mathrm{d}E} = \frac{1}{m}T$.

$$\boxed{T_0 \doteq \pi\sqrt{2m}\,\frac{1}{(\alpha^2 + 4\beta E_0)^{1/4}}}$$

$U = U_0|x|$ $\qquad$ $U'(x=0)$ ne obstaja



$U = U_0 x^4$: $\qquad$ $\underline{U''(x=0) = 0}$

Premočrtno gibanje pod vplivom nekonzervativne sile