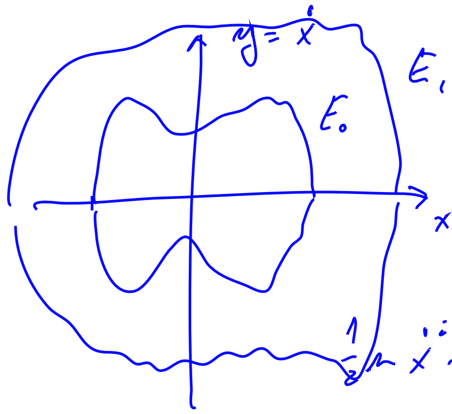
Predavanje 9. november 2020

Fazni portret



$$\frac{1}{2} m \dot{x}^2 + V(x) = E_0$$

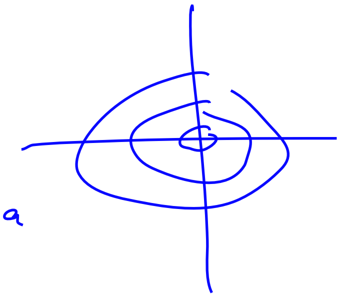
$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{\ddot{x}}{\dot{x}}$$

$$y = \dot{x} = \pm\sqrt{\frac{2}{m}\left(E_0 - V(x)\right)}$$

graf seka os x, ko je $E_0 = V(x) \iff \dot{x} = 0$

$$m\ddot{x} = f = -\frac{dV}{dx}$$

Primer: harmonični oscilator $V(x) = \frac{1}{2} k x^2$



$$\frac{1}{2} m y^2 + \frac{1}{2} k x^2 = E_0 \qquad \text{elipsa}$$

$$\frac{x^2}{\left(\sqrt{\frac{2E_0}{k}}\right)^2} + \frac{y^2}{\left(\sqrt{\frac{2E_0}{m}}\right)^2} = 1$$

$$a = \sqrt{\frac{2E_0}{k}}, \quad b = \sqrt{\frac{2E_0}{m}} \qquad A = \pi a b = 2\pi\sqrt{\frac{1}{mk}}\, E_0$$

↑ ploščina faznega portreta

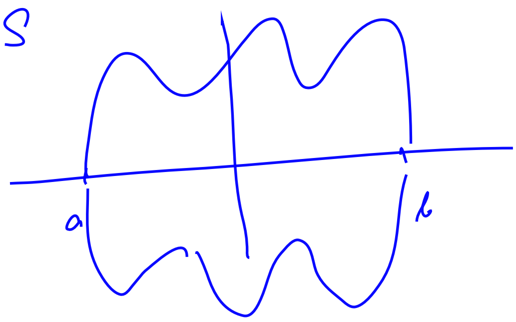
$$T = 2\pi\sqrt{\frac{m}{k}}$$

$$A = \frac{1}{m} T E_0 \qquad \frac{dA}{dE_0} = \frac{1}{m} T$$

.

Trditev 1. *Naj bo S ploščina faznega portreta periodičnega gibanja. Potem* $\frac{\mathrm{d}S}{\mathrm{d}E} = \frac{1}{m}T$.

S



$$S = 2\int_{a(E)}^{b(E)} \sqrt{\frac{2}{m}\left(E - V(x)\right)}\, dx$$

$$\frac{dS}{dE} = b'(E)\, 2\sqrt{\frac{2}{m}}\underbrace{\sqrt{E-U(b)}}_{=0} - a'(E)\, 2\sqrt{\frac{2}{m}}\underbrace{\sqrt{E-U(a)}}_{=0} + 2\sqrt{\frac{2}{m}}\,\frac{1}{2}\int_a^b \frac{dx}{\sqrt{E-U(x)}}$$

$$\frac{dS}{dE} = \sqrt{\frac{2}{m}}\int_a^b \frac{dx}{\sqrt{E-U(x)}} = \frac{1}{m}T$$

Premočrtno gibanje pod vplivom nekonzervativne sile

$$\left[m\ddot{x} = -\frac{dV}{dx} + \left(-\frac{\dot{x}}{|\dot{x}|}g(\dot{x})\right) \right] \dot{x} \qquad g(\dot{x}) > 0$$

$$m\dot{x}\ddot{x} + \dot{x}\frac{dV}{dx} = -\frac{\dot{x}^2}{|\dot{x}|}g(\dot{x}) = -|\dot{x}|g(\dot{x})$$

$$\frac{d}{dt}\left(\frac{1}{2}m\dot{x}^2 + U(x)\right) = -|\dot{x}|g(\dot{x})$$

$$\frac{dE}{dt} = -|\dot{x}|g(\dot{x}) < 0$$

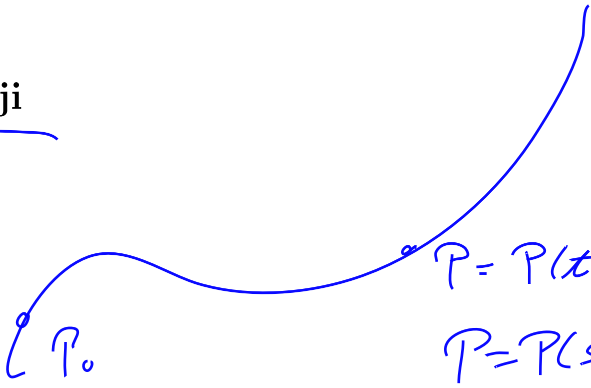
E monotono pada

$$E = -\int_{t_0}^{t} |\dot{x}|g(\dot{x})\,dt + E_0$$

Tipični primer: dušeno nihanje

$$m\ddot{x} = -kx - \ell\dot{x} \quad + A_0 \sin\omega t$$

Gibanje po krivulji



$\vec{r} = \vec{r}(s)$

$P = P(t)$

$P = P(s(t))$

$s = s(t)$ dolžina gibanja po krivulji

$$\vec{v} = \frac{d\vec{r}}{dt} = \underbrace{\frac{d\vec{r}}{ds}}_{\vec{e}_t}\frac{ds}{dt}$$

$$\vec{v} = \dot{s}\,\vec{e}_t \quad \Big| \quad v = |\vec{v}| = |\dot{s}|$$

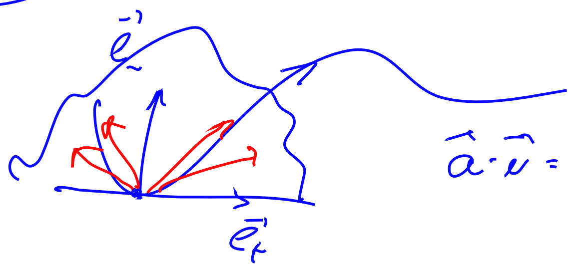
$$\vec{a} = \frac{d\vec{v}}{dt} = \ddot{s}\,\vec{e}_t + \dot{s}\,\underbrace{\frac{d\vec{e}_t}{ds}}_{=\kappa\vec{e}_n}\dot{s}$$

$\kappa \vec{e}_n \;:\; \kappa = \frac{1}{R}\;;\; R$ krivinski polmer

Kinematika gibanja po krivulji

Tangencialni pospešek, normalni pospešek; $\boxed{\vec{a} = \ddot{s}\vec{e}_t + \kappa\dot{s}^2\vec{e}_n}$.

$$\vec{a} = \underbrace{\ddot{s}\,\vec{e}_t}_{\text{tangentni pospešek}} + \kappa\dot{s}^2\,\vec{e}_n$$ normalni pospešek



$\vec{a}\cdot\vec{v} = \dot{s}\cdot\ddot{s}$

$\vec{a}$ leži v pritisnjeni ravnini in kaže v smer zavijanja krivulje.

aktivne sile

Newtonova enačba

$$m(\ddot{s}\vec{e}_t + \kappa \dot{s}^2 \vec{e}_n) = \vec{F}_r = \vec{F} + \boxed{\vec{S}}$$

sila vezi

sila vezi je idealna

$\Leftrightarrow$

Gibanje po gladki krivulji $\Leftrightarrow$ $\vec{S}\cdot\vec{e}_t = 0$ $\qquad \vec{S} \perp \vec{e}_t$

$$A(\vec{S}) = \int_{P_1}^{P_2} \vec{S}\cdot d\vec{r} = \int_{t_1}^{t_2} \vec{S}\cdot \dot{s}\vec{e}_t\, dt = 0$$

Sila vezi je idealna, če je $\vec{S} \perp \vec{e}_t$ $\Leftrightarrow$ delo sile vezi je enako nič.

Sila vezi, idealna vez ($\vec{S}\cdot\vec{e}_t = 0$).

Trditev 2. *Delo idealne sile vezi je enako nič.*

$\vec{e}_b = \vec{e}_t \times \vec{e}_n$

Coulombovo zakon trenja

$\vec{S} = S_1\vec{e}_t + S_2\vec{e}_n + S_3\vec{e}_b$

$S_1 = S_1(S_2, S_3)$

$\vec{S}\cdot\vec{v} = \dot{s}S_1 < 0$ $\qquad \dot{s}S_1 = \text{konst}$

$m\vec{a} = \vec{F} + \vec{S}$

$$\left.\begin{aligned} m\ddot{s} &= \vec{F}\cdot\vec{e}_t + \vec{S}\cdot\vec{e}_t \\ m\kappa\dot{s}^2 &= \vec{F}\cdot\vec{e}_n + \vec{S}\cdot\vec{e}_n \\ 0 &= \vec{F}\cdot\vec{e}_b + \vec{S}\cdot\vec{e}_b \end{aligned}\right\}$$

$S_1 = S_1(S_2, S_3)$ $\qquad \boxed{s, S_2}$

$\Rightarrow \vec{S}\cdot\vec{e}_b = \vec{F}\cdot\vec{e}_b$

Če je krivulja gladka

$m\ddot{s} = \vec{F}\cdot\vec{e}_t$

Redukcija na premočrtno gibanje

Gibanje po gladki krivulji je reducibilno na premočrtno gibanje v konfiguracijskem prostoru ločne dolžine.

$m\ddot{s} = \vec{F}\cdot\vec{e}_t$

Ali je konzervativna?

Gibanje po gladki krivulji pod vplivom konzervativne sile je integrabilno.

$\vec{F} = -\mathrm{grad}\, V(\vec{r})$

Ali je tudi $\vec{F}\cdot\vec{e}_t$ potencialna?

$s \longmapsto \vec{r}(s) \longmapsto V(\vec{r}(s))$

$V(s) = V(\vec{r}(s))$

$$\frac{dV}{ds} = \frac{\partial V}{\partial x}\frac{dx}{ds} + \frac{\partial V}{\partial y}\frac{dy}{ds} + \frac{\partial V}{\partial z}\frac{dz}{ds} = \mathrm{grad}\, V\cdot\vec{e}_t = -\vec{F}\cdot\vec{e}_t$$

$$\boxed{m\ddot{s} = -\frac{dV}{ds}}$$

Matematično nihalo

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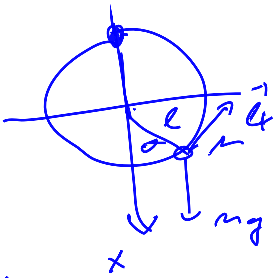
gibanje po gladki krožnici pod vplivom sile teže



$\boxed{s = l\vartheta}$

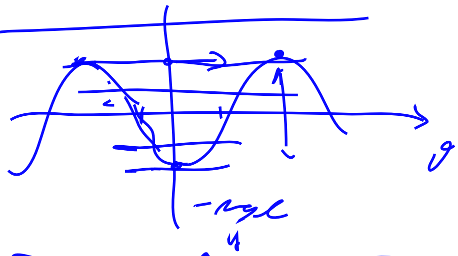
$V = V(s)$?

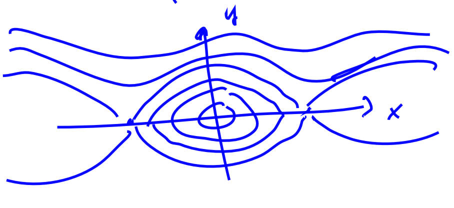
$V = -mgx = -mgl\cos\vartheta$

$$\boxed{V = -mgl\cos\frac{s}{l}}$$

- aproksimacija majhnega nihanja;
- harmonična aproksimacija;
- libracijska $T = 2\pi\sqrt{l/g}\sqrt{\theta_0/\sin\theta_0}$.
- ocena nihajnega časa matematičnega nihala:

$$2\pi\sqrt{\frac{l}{g}} \le T \le \frac{1}{\cos\frac{\theta_0}{2}}2\pi\sqrt{\frac{l}{g}}.$$





$$T = \sqrt{2m}\int_{-s_0}^{s_0}\frac{ds}{\sqrt{E_0 + mgl\cos\frac{s}{l}}}$$

$E_0 = -mgl\cos\frac{s_0}{l}$

$s = \vartheta l$

$$T = 2\sqrt{2m}\int_0^{\vartheta_0}\frac{l\,d\vartheta}{\sqrt{mgl}\sqrt{\cos\vartheta-\cos\vartheta_0}} = 2\sqrt{2}\sqrt{\frac{l}{g}}\int_0^{\vartheta_0}\frac{d\vartheta}{\sqrt{\cos\vartheta-\cos\vartheta_0}} =$$

$$\cos\vartheta = \cos^2\frac{\vartheta}{2} - \sin^2\frac{\vartheta}{2}$$

$$1 = \cos^2\frac{\vartheta}{2} + \sin^2\frac{\vartheta}{2} \implies 1-\cos\vartheta = 2\sin^2\frac{\vartheta}{2}$$

$$\cos\vartheta = \underline{1} - 2\sin^2\frac{\vartheta}{2}$$

$$= 2\sqrt{\frac{l}{g}}\int_0^{\vartheta_0}\frac{d\vartheta}{\underline{\sqrt{\sin^2\frac{\vartheta_0}{2}-\sin^2\frac{\vartheta}{2}}}}$$

$$\underline{\sin\frac{\vartheta}{2}} = \underline{\sin\frac{\vartheta_0}{2}}\,\underline{u} \leftarrow$$

$$\frac{1}{2}\cos\frac{\vartheta}{2}\,d\vartheta = \sin\frac{\vartheta_0}{2}\,du$$

$$= 4\sqrt{\frac{l}{g}}\int_0^1\frac{\cancel{\sin\frac{\vartheta_0}{2}}\,du}{\sqrt{1-\sin^2\frac{\vartheta_0}{2}u^2}\;\cancel{\sin\frac{\vartheta_0}{2}}\sqrt{1-u^2}} \qquad d\vartheta = \frac{2\sin\frac{\vartheta_0}{2}\,du}{\sqrt{1-\sin^2\frac{\vartheta_0}{2}u^2}}$$

$$= 4\sqrt{\frac{l}{g}}\int_0^1\frac{du}{\sqrt{\left(1-\sin^2\frac{\vartheta_0}{2}u^2\right)(1-u^2)}} = 4\sqrt{\frac{l}{g}}\,K\left(\sin\frac{\vartheta_0}{2}\right)$$

popolni eliptični integral prve vrste

$$\cos^2\frac{\vartheta_0}{2} = 1-\sin^2\frac{\vartheta_0}{2} \leq \underline{1-\sin^2\frac{\vartheta_0}{2}u^2} < 1$$

$$4\sqrt{\frac{l}{g}}\int_0^1\frac{du}{\sqrt{1-u^2}} < T < 4\sqrt{\frac{l}{g}}\,\frac{1}{\cos\frac{\vartheta_0}{2}}\underbrace{\int_0^1\frac{du}{\sqrt{1-u^2}}}_{\arcsin\big|_0^1 = \frac{\pi}{2}}$$

$$\boxed{2\pi\sqrt{\frac{l}{g}} < T < \frac{1}{\cos\frac{\vartheta_0}{2}}\,2\pi\sqrt{\frac{l}{g}}}$$

$$m\vec{a} = \vec{F}\ /\cdot\vec{e}_t\,, \qquad \vec{e}_t = \vec{e}_\vartheta = -\sin\vartheta\,\vec{i} + \cos\vartheta\,\vec{j}$$

$$\vec{a}\cdot\vec{e}_\vartheta = \underline{l\ddot{\vartheta}} + 2\dot{l}\dot{\vartheta} \qquad m l\ddot{\vartheta} = \vec{F}\cdot\vec{e}_\vartheta = -mg\sin\vartheta$$

$$\vec{F} = mg\,\vec{i}$$

$$\ddot{\vartheta} + \frac{g}{l}\sin\vartheta = 0 \qquad |\vartheta| \ll 1 \implies \underline{\sin\vartheta \doteq \vartheta}$$

$$\ddot{\vartheta} + \frac{g}{l}\vartheta = 0 \implies \vartheta = A\cos\sqrt{\frac{g}{l}}\,t + B\sin\sqrt{\frac{g}{l}}\,t \implies T = 2\pi\sqrt{\frac{l}{g}}$$