Predavanje 16. november 2020

Matematično nihalo

$$U = -mgl\cos\vartheta = -mgl\cos\frac{s}{l}$$

- aproksimacija majhnega nihanja; ✓
- harmonična aproksimacija;
- libracijska $T = 2\pi\sqrt{l/g}\sqrt{\theta_0/\sin\theta_0}$. ✓
- ocena nihajnega časa matematičnega nihala:

$$2\pi\sqrt{\frac{l}{g}} \le T \le \frac{1}{\cos\frac{\theta_0}{2}} 2\pi\sqrt{\frac{l}{g}}. \checkmark$$

$$T \doteq \pi\sqrt{\frac{m}{2}}\left(\frac{1}{\sqrt{\chi(a)}} + \frac{1}{\sqrt{\chi(b)}}\right) = \pi\sqrt{2m}\,\frac{1}{\sqrt{\chi(s_0)}}$$

$$E_0 - U(s) = (s_0 - s)(s_0 + s)\,\chi(s) \implies \chi(s) = \frac{E_0 - U(s)}{s_0^2 - s^2}$$

$$\chi(s_0) = \lim_{s\to s_0} \frac{-mgl\cos\frac{s_0}{l} + mgl\cos\frac{s}{l}}{s_0^2 - s^2} = \frac{mgl\left(-\sin\frac{s_0}{l}\right)}{-2s_0 l} = mg\frac{\sin\frac{s_0}{l}}{2s_0}$$

$$\chi(\vartheta_0) = \frac{1}{2} mg \frac{1}{l}\frac{\sin\vartheta_0}{\vartheta_0}$$

$$T = \pi\sqrt{2m}\frac{\sqrt{2l\vartheta_0}}{\sqrt{mg\sin\vartheta_0}} = 2\pi\sqrt{\frac{l}{g}}\sqrt{\frac{\vartheta_0}{\sin\vartheta_0}}$$

$$T_0 = \underbrace{2\pi\sqrt{\frac{m}{U''(s=0)}}}$$

$$U = -m\vec{g}\cdot\vec{r}(s)$$

$$U'(s) = \operatorname{grad} U \cdot \frac{d\vec{r}}{ds} = \operatorname{grad} U\, \underline{\vec{e}_t}$$

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$$U''(s) = \vec{e}_t \cdot \underbrace{\frac{\partial^2 U}{\partial \vec{r}^2}} \cdot \vec{e}_t + \operatorname{grad} U \cdot \varkappa\, \vec{e}_n$$

$U = -m\vec{g}\cdot\vec{r}$

$\operatorname{grad} U = -m\vec{g} \Rightarrow \dfrac{\partial^2 U}{\partial \vec{r}^2} = 0 \; ; \; U''(s) = -m\vec{g}\cdot\vec{e}_n \varkappa$

$U'(s=0) = 0 \iff \operatorname{grad} U(s=0)\cdot\vec{e}_t = 0 \Rightarrow \operatorname{grad} U(s=0) = \alpha\vec{e}_n + \beta\vec{e}_b$

Harmonična aproksimacija periode gibanja po krivulji.

$U''(s) = \vec{e}_t\cdot\dfrac{\partial^2 U}{\partial \vec{r}^2}\cdot\vec{e}_t + \alpha\varkappa$

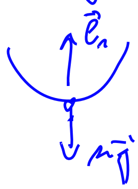
$$T = 2\pi\sqrt{1/gf''(x_0)}.$$

Primer $U = -m\vec{g}\cdot\vec{r} \Rightarrow U''(s=0) = -m\vec{g}\cdot\vec{e}_n\varkappa$

$\operatorname{grad} U = -m\vec{g} = \alpha\vec{e}_n + \beta\vec{e}_b \Rightarrow \alpha = -m\vec{g}\cdot\vec{e}_n$

Ravninska krivulja

$\operatorname{grad} U\cdot\vec{e}_t = 0 \Rightarrow |\vec{g}\cdot\vec{e}_n| = g$



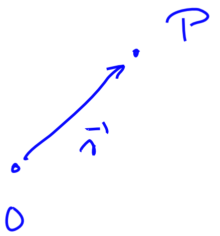
$U''(s=0) = mg\varkappa$

$T_0 = 2\pi\sqrt{\dfrac{m}{mg\varkappa}} = \dfrac{2\pi}{\sqrt{g\varkappa}} = 2\pi\sqrt{\dfrac{\ell}{g}}$

Gibanje v polju centralne sile

$y = f(x) \qquad \varkappa = |f''(x=0)| \qquad \varkappa = \dfrac{1}{\ell}$

Definicija 1. *Sila $\vec{F}$ je centralna, če obstaja točka O (center sile) tako, da velja $\vec{F} = f(|P-O|)(P-O)/|P-O|$.*



$\vec{r} = P - O \qquad r = |\vec{r}|$

$\dfrac{\vec{r}}{|\vec{r}|} = \dfrac{P-O}{|P-O|}$

$\vec{F} = f(|P-O|)\dfrac{P-O}{|P-O|} = f(r)\dfrac{\vec{r}}{r} = f(r)\vec{e}_r$

$\vec{e}_r$

$$\vec{F} = f(r)\,\vec{e}_r \qquad U = -\int_{r_0}^{r} f(r)\,dr + U_0$$

$$+\operatorname{grad} U = -\frac{\partial U}{\partial \vec{r}} = -\frac{\partial U}{\partial r}\frac{\partial r}{\partial \vec{r}} = -f(r)\frac{\partial r}{\partial \vec{r}}$$

$$\frac{\partial r}{\partial \vec{r}} = \operatorname{grad} r = \frac{\partial r}{\partial x}\vec{i} + \frac{\partial r}{\partial y}\vec{j} + \frac{\partial r}{\partial z}\vec{k} = \frac{1}{r}(x\vec{i} + y\vec{j} + z\vec{k}) = \frac{\vec{r}}{r} = \vec{e}_r$$

$$\frac{\partial r}{\partial x} = \frac{\partial}{\partial x}\sqrt{x^2+y^2+z^2} = \frac{2x}{2r} = \frac{x}{r}$$

$$+\operatorname{grad} U = -f(r)\,\vec{e}_r = -\vec{F}$$

$$T + U = E_0$$

Če je funkcija $f = f(r)$ zvezna, je centralna sila konzervativna.

$$\dot{\vec{r}} = \dot{P}$$

vrtilna količina točke P glede na pol O

$$\vec{l} = (P - O) \times m\dot{\vec{r}} \qquad \vec{l}(O,P)$$

Trditev 1. *Pri gibanju v polju centralnih sil je vrtilna količina $\vec{l}(O,P)$ konstantna.*

$$\vec{l} = \vec{r} \times m\vec{v} = \vec{r} \times m\dot{\vec{r}}$$

$$\dot{\vec{l}} = \dot{\vec{r}} \times m\dot{\vec{r}} + \vec{r} \times m\ddot{\vec{r}} = \vec{r} \times \vec{F} = \vec{r} \times f(r)\frac{\vec{r}}{r} = \vec{0} \qquad \vec{l} = \vec{l}_0$$

Trditev 2. *Gibanje v polju centralnih sil je ravninsko. Točka se giblje po ravnini, ki vsebuje center sil in ima normalo v smeri vektorja vrtilne količine.*

Dokaz

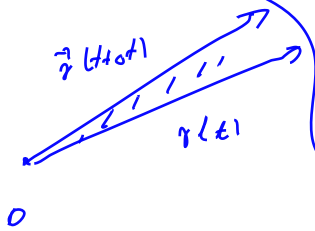
$$(\vec{r} - \vec{r}_0)\cdot\vec{n} = 0$$

$$\vec{r} = P - O$$

$$(P - O)\cdot\vec{l} = (P - O)\cdot\big((P - O)\times m\dot{P}\big) = 0$$

$$(P - O)\cdot\vec{l}_0 = 0 \qquad (\vec{l} = \vec{l}_0)$$

$$\vec{r}\cdot\vec{l}_0 = 0 \qquad \vec{r}_0 = \vec{0}.$$



$$\Delta \vec A = \frac12 \vec r(t) \times \vec r(t+\Delta t) = \frac12 \vec r \times \dot{\vec r}\,\Delta t + \cancel{o(\Delta t)}$$

$$\vec r(t+\Delta t) = \vec r(t) + \dot{\vec r}(t)\,\Delta t + o(\Delta t)$$

$$\lim_{\Delta t\to 0} \frac{\Delta \vec A}{\Delta t}$$ ploščinska hitrost

$$\dot{\vec A} = \frac12 \vec r\times \dot{\vec r} = \frac12 r^2\dot\varphi\, \vec e_r \times \vec e_\varphi = \frac12 r^2 \dot\varphi\, \vec k$$

$$\vec r = r\,\vec e_r, \qquad \vec v = \dot{\vec r} = \dot r\, \vec e_r + r\dot\varphi\, \vec e_\varphi$$

$$A = \frac12 r^2\dot\varphi \qquad C_0 = r^2\dot\varphi$$

Ploščinska hitrost, zveza $l = \left|\vec l\right| = mC_0$, kjer je C_0 dvojna ploščinska hitrost.

$$\vec l = \vec r \times m\dot{\vec r} \qquad l = m r^2\dot\varphi = mC_0 \qquad \boxed{l = mC_0}$$

$$l = |\vec l| \ge 0 \qquad l = m r^2 \dot\varphi$$

Izrek 1. *Gibanje v polju centralne sile ima dve konstanti gibanja, energijo in ploščinsko hitrost.*

Keplerjevi zakoni

Zgodovinski oris: Tycho Brahe (1546-1601), Johannes Kepler (1571-1630), Isaac Newton (1643-1727);

Trditev 3. *Če je ploščinska hitrost konstantna, je obodni pospešek enak nič.*

$$a_\varphi = r\ddot\varphi + 2\dot r\dot\varphi = \frac1r\left(r^2\ddot\varphi + 2r\dot r\dot\varphi\right) = \frac1r\left(r^2\dot\varphi\right)^{\cdot} = 0$$

$$(\dot\varphi)' \qquad (r^2)' \qquad C_0$$

Binetova formula

$$a_r = -C_0^2u^2\left(d^2u/d\theta^2 + u\right).$$

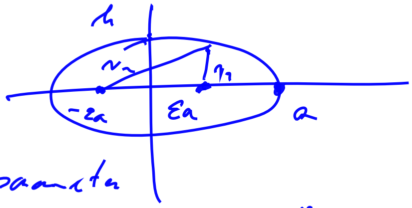
$$a_r = \ddot r - r\dot\varphi^2 \qquad r^2\dot\varphi = C_0 \qquad u = \frac1r$$

$$\dot r = \frac{dr}{dt} = \frac{dr}{d\varphi}\frac{d\varphi}{dt} = r'\dot\varphi = r'\frac{C_0}{r^2} = -C_0\left(\frac1r\right)' = -C_0u'$$

$$r = r(\varphi);\quad \varphi = \varphi(t) \qquad \ddot r = -C_0\frac{du'}{dt} = -C_0u''\dot\varphi = -C_0^2u^2u''$$

$$a_r = -C_0^2u^2u'' - r\frac{C_0^2}{r^4} = -C_0^2u^2\left(u + u''\right)$$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$



$$r_1+r_2=\text{konst.}=2a$$

goriščni parameter

$$\boxed{r=\frac{p}{1+\varepsilon\cos\varphi}}$$

$$r_1=a-\varepsilon a$$

$$r_2=a+\varepsilon a$$

$$m a_r = F_r$$

$$u=\frac{1}{r}=\frac{1}{p}(1+\varepsilon\cos\varphi)$$

$$u'=-\frac{\varepsilon}{p}\sin\varphi\ ;\quad u''=-\frac{\varepsilon}{p}\cos\varphi$$

$$a_r=-c_0^2u^2(u+u'')=-c_0^2\frac{1}{r^2}\frac{1}{p}$$

$$a_r=\left(-\frac{c_0^2}{p}\right)\frac{1}{r^2}$$

Trditev 4. *Če velja prvi Keplerjev zakon, je radialni pospešek obratno sorazmeren kvadratu oddaljenosti do centra sil.*

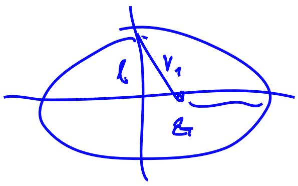
$$\vec F=-\kappa\frac{mM}{r^2}\vec e_r \quad / \quad \boxed{\frac{c_0^2}{p}=\kappa M}$$

$$\boxed{T^2=ka^3} \Rightarrow \frac{c_0^2}{p}$$ je enak za vse planete v osončju

$$A=T\cdot\frac{1}{2}c_0$$

$$A=\pi ab$$

$$r=\frac{p}{1+\varepsilon\cos\varphi}$$



$$a(1-\varepsilon)=a-\varepsilon a=\frac{p}{1+\varepsilon} \Rightarrow a=\frac{p}{1-\varepsilon^2}$$

$$a^2=r_1^2=b^2+\varepsilon^2a^2 \Rightarrow b^2=a^2(1-\varepsilon^2)$$

$$b=a\sqrt{1-\varepsilon^2}=\frac{p}{\sqrt{1-\varepsilon^2}}$$

$$\pi ab=\frac{1}{2}Tc_0$$

$$\pi^2a^2b^2=\frac{1}{4}T^2c_0^2=\frac{1}{4}ka^3c_0^2$$

$$\pi^2b^2=\frac{1}{4}kac_0^2$$

$$\pi^2\cdot\frac{p^2}{1-\varepsilon^2}=\frac{1}{4}k\cdot\frac{p}{1-\varepsilon^2}c_0^2 \Rightarrow \boxed{\frac{c_0^2}{p}=\frac{4\pi^2}{k}}$$

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