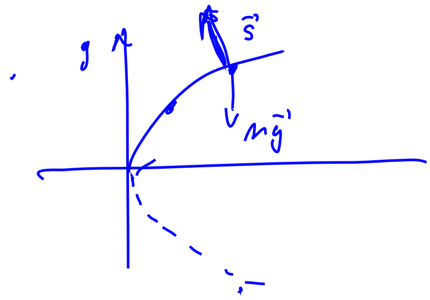


Vaje 17. november 2020

$p>0$

1. Točka se giblje brez trenja po paraboli $y^2 = 2px$ pod vplivom sile teže v negativni smeri osi y. Določi kje točka zapusti parabolo, če točka prične gibanje iz mirovanja pri $x = 2p$.

$$m\vec a = m\vec g + \vec S$$

$$\vec r = x\vec i + y\vec j = x\vec i + \sqrt{2px}\,\vec j$$

$$\dot{\vec r} = \dot x\vec i + \sqrt{2p}\,\frac{1}{2\sqrt x}\,\vec j\,\dot x$$

$$\ddot{\vec r} = \ddot x\vec i + \left(\frac{\sqrt p}{\sqrt{2x}}\,\ddot x - \sqrt{\frac p2}\,\frac12\,\frac{1}{x\sqrt x}\,\dot x^2\right)\vec j$$

$$\vec n \parallel -\sqrt{\frac{p}{2x}}\,\dot x\,\vec i + \dot x\,\vec j$$

$$\vec n = \frac{1}{\sqrt{1+p/2x}}\left(-\sqrt{\frac{p}{2x}}\,\vec i + \vec j\right); \quad \vec S = S\vec n$$

$$\vec S = S\left(-\sqrt{\frac{p}{2x}}\,\vec i + \vec j\right)$$

$$m\ddot x = -\sqrt{\frac{p}{2x}}\,S \qquad / \cdot \left(-\sqrt{\frac{p}{2x}}\right)$$

$$m\left(\sqrt{\frac{p}{2x}}\,\ddot x - \frac12\sqrt{\frac p2}\,\frac{1}{x\sqrt x}\,\dot x^2\right) = -mg + S$$

$$-\frac12\sqrt{\frac p2}\,\frac{\dot x^2}{x\sqrt x}\,m = -mg + S\left(1+\frac{p}{2x}\right)$$

Točka zapusti parabolo $\iff S=0$

$$\boxed{\frac12\sqrt{\frac p2}\,\frac{\dot x^2}{x\sqrt x} = g}$$

$$\frac12 m(\dot x^2 + \dot y^2) + mgy = E_0 = mg\sqrt{2p\cdot 2p} = 2mgp$$

$$\frac12 \cancel{m}\,\dot x^2\left(1+\frac{2p}{4x}\right) + \cancel{m}g\sqrt{2px} = 2\cancel{m}gp$$

$$\boxed{\dot x^2 = 2\,\frac{2gp - g\sqrt{2px}}{1+\frac{p}{2x}}}$$

$$\frac12\sqrt{\frac p2}\,2\left(2p - \sqrt{2px}\right) = x\sqrt x\left(1+\frac{p}{2x}\right)$$

$$2\sqrt{\frac p2}\,p - p\sqrt x = x\sqrt x + \frac p2\sqrt x$$

$\cancel{x=p}$

$$0 = x\sqrt x + \frac{3p}{2}\sqrt x - 2\sqrt{\frac p2}\,p \qquad \sqrt x = u\sqrt p$$

$$0 = p\sqrt p\,u^3 + \frac{3p}{2}\sqrt p\,u - 2\sqrt{\frac p2}\,p$$

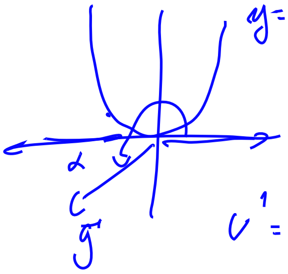
$$0 = u^3 + \frac32 u - \sqrt2 \;;$$

$u = \sqrt2$; $\quad 2\sqrt2 + \frac32\sqrt2 - \sqrt2 \neq 0$

$u = \frac{1}{\sqrt2}$: $\quad \frac{1}{2\sqrt2} + \frac{3}{2\sqrt2} - \sqrt2 = \frac{2}{\sqrt2} - \sqrt2 = 0$

$$\boxed{x = \frac p2}$$

2. Točka se giblje paraboli $y = \frac{1}{2}px^2$ pod vplivom sile teže $m\vec{g} = mg(\cos\alpha\vec{\imath} + \sin\alpha\vec{\jmath})$. Določi periodo majhnega nihanja okoli ravnovesne lege.



$y = \frac{1}{2}px^2$; $p>0$ $\qquad T = 2\pi\sqrt{\frac{m}{U''(s_0)}}$

$U = -m\vec g\cdot\vec r$

$U' = \frac{dU}{ds} = -m\vec g\cdot\frac{d\vec r}{ds} = -m\vec g\cdot\vec e_t$; $0 = U'(s_0) = -m\vec g\cdot\vec e_t$

$\vec g \perp \vec e_t$

$U'' = -m\vec g\cdot\varkappa\,\vec e_n$

$\vec e_t = \sin\alpha\,\vec i - \cos\alpha\,\vec j$

$U'' = -mg(\cos\alpha\,\vec i + \sin\alpha\,\vec j)\cdot\varkappa\cdot\mathrm{sg}\cdot(\cos\alpha\,\vec i + \sin\alpha\,\vec j)$ $\qquad \vec e_n \perp \vec e_t \Rightarrow \vec e_n \parallel \vec g$

$= -mg\,\mathrm{sg}\,\varkappa$; $\mathrm{sg} = -1$ $\qquad \vec e_n = (\cos\alpha\,\vec i + \sin\alpha\,\vec j)\,\mathrm{sg}$

$\vec e_n = -\cos\alpha\,\vec i - \sin\alpha\,\vec j$ $\qquad \mathrm{sg} = \pm 1$

$T = 2\pi\sqrt{\frac{m}{mg\varkappa}} = 2\pi\sqrt{\frac{1}{g\varkappa}}$ $\qquad \varkappa = \left|\frac{y''x' - y'x''}{(x'^2+y'^2)^{3/2}}\right| = \left|\frac{y''}{(1+y'^2)^{3/2}}\right| = \frac{p}{(1+p^2x^2)^{3/2}}$

$0 = m\vec g\cdot\vec e_t$ $\qquad (\cos\alpha\,\vec i + \sin\alpha\,\vec j)\cdot(\vec i + px\,\vec j) = 0$

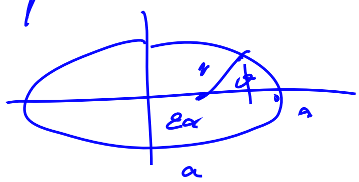
$\cos\alpha + \sin\alpha\,px = 0 \Rightarrow px = -\frac{\cos\alpha}{\sin\alpha}$

$\varkappa = \frac{p}{\left(1 + \frac{\cos^2\alpha}{\sin^2\alpha}\right)^{3/2}} = \frac{p}{\frac{1}{\sin^3\alpha}} = p\sin^3\alpha$

$T = 2\pi\frac{1}{\sin\alpha}\sqrt{\frac{1}{gp\sin\alpha}}$

3. Izpelji polarno obliko enačbe elipse.

$$r = \frac{p}{1+\varepsilon\cos\varphi}$$



$$\boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1}$$

$$a = \frac{p}{1-\varepsilon^2} \Rightarrow a = a\varepsilon^2 + p$$

$$b = \frac{p}{\sqrt{1-\varepsilon^2}} \leftarrow$$

$$x = \varepsilon a + r\cos\varphi = \varepsilon a + \frac{p\cos\varphi}{1+\varepsilon\cos\varphi} = \frac{\varepsilon a + \overbrace{(\varepsilon^2 a + p)}^{a}\cos\varphi}{1+\varepsilon\cos\varphi}$$

$$y = r\sin\varphi = \frac{p\sin\varphi}{1+\varepsilon\cos\varphi}$$

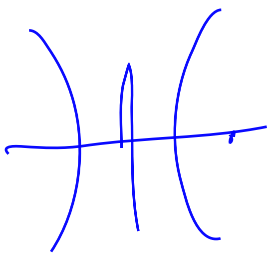
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{1}{(1+\varepsilon\cos\varphi)^2}\left(\left(\frac{a(\varepsilon+\cos\varphi)}{a}\right)^2 + \left(\frac{p\sin\varphi}{b}\right)^2\right) =$$

za elipso

$$= \frac{1}{(1+\varepsilon\cos\varphi)^2}\left(\varepsilon^2 + 2\varepsilon\cos\varphi + \cos^2\varphi + (1-\varepsilon^2)\sin^2\varphi\right)$$

$$= \frac{1}{(1+\varepsilon\cos\varphi)^2}\left(1 + 2\varepsilon\cos\varphi + \underbrace{\varepsilon^2(1-\sin^2\varphi)}_{\varepsilon^2\cos^2\varphi}\right) = 1$$

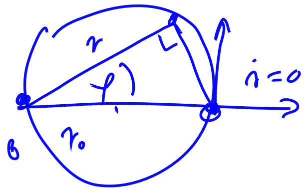
$$r = \frac{p}{1+\varepsilon\cos\varphi}$$



$0 \le \varepsilon < 1$ elipsa

$\varepsilon = 1$ parabola

$\varepsilon > 1$ hiperbola $\quad \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$



4. Točka se giblje v polju centralne sile po krožnici s centrom sile na krožnici.

   (a) Določi silo.

   (b) Skiciraj pripadajoči efektivni potencial.

   (c) Izračunaj čas prihoda v center sile,če je v začetnem trenutku točka diametralno od centra sile. Izračunaj pripadajočo energijo.

$$\cos\varphi = \frac{r}{2r_0}$$

$$r = 2r_0 \cos\varphi \qquad a_r = -C_0^2 u^2 (u + u'') \; ; \quad u = \frac{1}{r}$$

$$u = \frac{1}{2r_0 \cos\varphi} = \frac{1}{2r_0}\cos^{-1}\varphi \Rightarrow \boxed{\cos^{-1}\varphi = 2r_0 \frac{1}{r}}$$

$$u' = \frac{1}{2r_0}\left(-\cos^{-2}\varphi(-\sin\varphi)\right) = \frac{1}{2r_0}\sin\varphi\cos^{-2}\varphi$$

$$u'' = \frac{1}{2r_0}\left(\cos\varphi\cdot\cos^{-2}\varphi + \sin\varphi(-2\cos^{-3}\varphi)(-\sin\varphi)\right) =$$

$$= \frac{1}{2r_0}\left(\cos^{-1}\varphi + 2\underbrace{\sin^2\varphi}_{1-\cos^2\varphi}\cos^{-3}\varphi\right) = \frac{1}{2r_0}\left(-\cos^{-1}\varphi + 2\cos^{-3}\varphi\right)$$

$$u + u'' = \frac{1}{2r_0}\left(\cos^{-1}\varphi - \cos^{-1}\varphi + 2\cos^{-3}\varphi\right) = \frac{1}{r_0}\cos^{-3}\varphi = \frac{1}{r_0} 8r_0^3\frac{1}{r^3}$$

$$a_r = -C_0^2\frac{1}{r^2}8r_0^2\cdot\frac{1}{r^3} = -8r_0^2C_0^2\frac{1}{r^5}$$

$$\vec{F} = m a_r \vec{e}_r = -8mr_0^2C_0^2\frac{1}{r^5}\vec{e}_r \qquad f(r) = -8mr_0^2C_0^2\frac{1}{r^5}$$

$$\vec{F} = -\mathrm{grad}\, V \qquad V = -\int_{r_1}^{r} f(r)\,dr = -2mr_0^2C_0^2\frac{1}{r^4}$$

$$T + V = E_0$$

$$\frac{1}{2}mv^2 + V = E_0 \Rightarrow \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\vartheta}^2\right) + V = E_0$$

$$\vec{v} = \dot{r}\vec{e}_r + r\dot{\vartheta}\vec{e}_\vartheta \qquad C_0 = r^2\dot{\vartheta} \Rightarrow \dot{\vartheta} = \frac{C_0}{r^2}$$

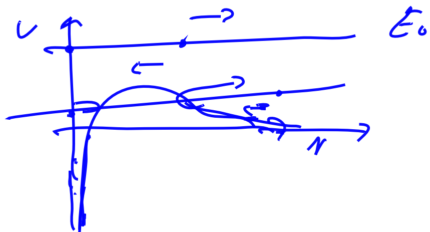
$$\boxed{\frac{1}{2}m\dot{r}^2 + \frac{1}{2}m\frac{C_0^2}{r^2} + V = E_0}$$

efektivni potencial

$$U = \frac{mC_0^2}{2r^2} + V(r)$$

$$l = mC_0$$

$$U = \frac{l^2}{2mr^2} - 2\frac{1}{m}r_0^2l^2\frac{1}{r^4}$$



$\frac{1}{2} c_0 \cdot T = A = \frac{1}{2} \pi r_0^2 \Rightarrow T = \frac{\pi r_0^2}{c_0}$

čas pada v center sile

$V = \frac{\ell^2}{2m r_0^2} - 2 \frac{1}{m} \ell^2 \frac{1}{r_0^2} = -\frac{3}{2} \frac{\ell^2}{m r_0^2}$

$\frac{1}{2} m \dot{r}^2 = 0 \qquad E_0 = -\frac{3}{2} \frac{\ell^2}{m r_0^2}$

5. Točka se giblje v polju centralne sile po tiru $r = r_0 \cos n\varphi$, $n \in \mathbb{N}$.
   (a) Določi silo.
   (b) Skiciraj pripadajoči efektivni potencial.
   (c) Določi čas, da iz točke največje oddaljenosti pade v center sile.