Predavanje 23. november 2020

$$\vec{v} = \dot{r}\,\vec{e}_r + r\dot{\vartheta}\,\vec{e}_\vartheta$$

$$v^2 = \vec{v}\cdot\vec{v} = \dot{r}^2 + r^2\dot{\vartheta}^2 = \dot{r}^2 + r^2\frac{C_0^2}{r^4}$$

Redukcija na premočrtno gibanje

$$E_0 = T + V = \frac{1}{2}mv^2 + V(r) = \underbrace{\frac{1}{2}m\dot{r}^2 + \frac{1}{2}m\frac{C_0^2}{r^2} + V(r)}$$

$$C_0 = r^2\dot{\vartheta} \Rightarrow \dot{\vartheta} = \frac{C_0}{r^2} \qquad l = mC_0$$

$$\boxed{E_0 = \underbrace{\frac{1}{2}m\dot{r}^2} + \frac{l^2}{2mr^2} + V(r)} \qquad U(r) = \underbrace{\frac{l^2}{2mr^2}} + V(r)$$

Efektivni potencial $U(r) = \frac{l^2}{2mr^2} + V(r)$.

$$t - t_0 = \operatorname{sgn}\dot{r}\int_{r_0}^{r}\frac{dr}{\sqrt{\frac{2}{m}(E_0 - U(r))}}$$

Določitev trajektorije

$$t = t(r)$$

$$r = r(t) \qquad \vartheta(t) = ?$$

$$\frac{d\vartheta}{dt} = \frac{C_0}{r^2} \Rightarrow \int d\vartheta = \int\frac{C_0}{r^2(t)}dt \Rightarrow \vartheta - \vartheta_0 = \int_{t_0}^{t}\frac{C_0}{r^2(t)}dt$$

$$t \longmapsto (r(t), \vartheta(t)) \longrightarrow \vec{r}(t) = r(t)\,\vec{e}(\vartheta(t))$$

$$r = r(\vartheta) \qquad \frac{dr}{d\vartheta} = \frac{dr}{dt}\cdot\frac{dt}{d\vartheta} = \frac{\dot{r}}{\dot{\vartheta}}$$

$$\frac{dr}{d\vartheta} = \frac{r^2}{C_0}\operatorname{sgn}\dot{r}\sqrt{\frac{2}{m}(E_0 - U(r))}$$

$$d\vartheta = \operatorname{sgn}\dot{r}\frac{C_0\,dr}{r^2\sqrt{\frac{2}{m}(E_0 - U(r))}} = \operatorname{sgn}\dot{r}\frac{l}{\sqrt{2m}}\frac{dr}{r^2\sqrt{E_0 - U(r)}}$$

$$\vartheta - \vartheta_0 = \operatorname{sgn}\dot{r}\frac{l}{\sqrt{2m}}\int_{r_0}^{r}\frac{dr}{r^2\sqrt{E_0 - U(r)}} \qquad \vartheta = \vartheta(r)$$

$$r = r(\vartheta)$$

Določitev tira

$$\theta = \theta_0 + \operatorname{sgn} \dot{r} \frac{l}{\sqrt{2m}} \int_{r_0}^{r} \frac{dr}{r^2 \sqrt{E_0 - U(r)}}.$$

$$\vec F = -\frac{\varkappa m M}{r^2}\vec e_r = -\frac{\gamma}{r^2}\vec e_r = -f(r)\vec e_r$$

Gibanje v polju gravitacijske sile

Kvalitativna analiza.

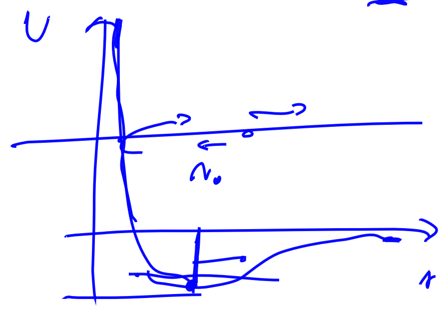
$$V - V_0 = -\int_{r_0}^{r} f(\rho)\,d\rho = \int_{r_0}^{r}\frac{\gamma}{\rho^2}\,d\rho = -\frac{\gamma}{r}+\frac{\gamma}{r_0}$$

$$f(\rho) = -\frac{\gamma}{\rho^2}$$

$$V(r) = -\frac{\gamma}{r}\ ;\quad \gamma>0$$

$$\lim_{r\to\infty} V(r) = 0$$

$$U = \frac{l^2}{2mr^2} + V = \frac{l^2}{2mr^2} - \frac{\gamma}{r}$$



$$U' = -\frac{l^2}{mr^3} + \frac{\gamma}{r^2} = \frac{1}{r^3}\left(-\frac{l^2}{m} + \gamma r\right)$$

$$U' = 0 \Rightarrow r_0 = \frac{l^2}{m\gamma}$$

$$U(r_0) = \frac{l^2 m^2\gamma^2}{2m\,l^4} - \frac{\gamma^2 m}{l^2} = -\frac{\gamma^2 m}{2l^2}$$

$E_0 = U_0 = -\frac{m\gamma^2}{2l^2}$ kroženje

$C_0 = r^2\dot\vartheta$; enakomerno kroženje

Tir gibanja.

$$\vartheta - \vartheta_0 = \operatorname{sgn}\dot r\,\frac{l}{\sqrt{2m}}\int_{r_0}^{r}\frac{dr}{r^2\sqrt{E_0 - \frac{l^2}{2mr^2} + \frac{\gamma}{r}}} = -\int_{u_0}^{u}\frac{du}{\sqrt{E_0 - u^2 + \frac{\gamma\sqrt{2m}}{l}u}}$$

$\dot r > 0$

$$u = \frac{1}{r}\frac{l}{\sqrt{2m}} \qquad du = -\frac{l}{\sqrt{2m}}\frac{1}{r^2}\,dr$$

$$E_0 - u^2 + \frac{\gamma\sqrt{2m}}{l}u = E_0 - \left(u - \frac{\gamma\sqrt{2m}}{2l}\right)^2 + \frac{\gamma^2 2m}{4l^2} =$$

$$= \underbrace{E_0 + \frac{\gamma^2 m}{2l^2}}_{} - \left(u - \frac{\gamma\sqrt m}{l\sqrt2}\right)^2 = \Delta^2\left(1 - \left(\frac{\ }{\Delta}\right)^2\right)$$

$$E_0 > U_0 = -\frac{\gamma^2 m}{2l^2} \quad \Big|\ \Delta^2 > 0$$

$$\vartheta - \vartheta_0 = \frac{\Delta}{\Delta}\arccos\left(\frac{u - \frac{\gamma\sqrt m}{l\sqrt2}}{\Delta}\right)\Bigg|_{u_0}^{u}$$

$$r_0 = \frac{l^2}{m\gamma}$$

$$\vartheta - \vartheta_0 = \arccos\frac{u - \frac{\gamma\sqrt m}{l\sqrt2}}{\Delta} - \arccos\frac{\frac{1}{r_0}\frac{l}{\sqrt{2m}} - \frac{\gamma\sqrt m}{l\sqrt2}}{\Delta}, \qquad \frac{\frac{1}{r_0}\frac{l}{\sqrt{2m}} - \frac{\gamma\sqrt m}{l\sqrt2}}{\Delta} = 0$$

$$r = \frac{p}{1+\epsilon\cos(\theta-\theta_0+\pi/2)}, \qquad p = \frac{l^2}{m\gamma}, \qquad \epsilon = \sqrt{1+\frac{2El^2}{m\gamma^2}}$$

$$r = \frac{p}{1+\varepsilon\cos\varphi}$$

$$\vartheta - \vartheta_0 = \arccos\frac{u - \frac{\gamma\sqrt m}{l\sqrt2}}{\Delta} - \frac{\pi}{2}$$

$$\Delta\cos\left(\vartheta - \vartheta_0 + \frac{\pi}{2}\right) + \frac{\gamma\sqrt m}{l\sqrt2} = u = \frac{1}{r} \Rightarrow r = \frac{1}{\frac{\gamma\sqrt m}{l\sqrt2} + \Delta\cos\left(\vartheta - \vartheta_0 + \frac{\pi}{2}\right)}$$

$$r = \frac{\frac{l\sqrt{2}}{\gamma\sqrt{m}}}{1 + \frac{l\sqrt{2}}{\gamma\sqrt{m}}\,\omega \cos(\vartheta - \vartheta_0 + \frac{\pi}{2})} =$$

$$\frac{l\sqrt{2}}{\gamma\sqrt{m}}\,\omega = \sqrt{\frac{l^2 2}{\gamma^2 m}\left(E_0 + \frac{\gamma^2 m}{2 l^2}\right)} = \sqrt{1 + \frac{2 l^2 E_0}{m \gamma^2}} = \varepsilon$$

Gibanje v polju potenciala $V = \alpha r^p$

$$r = \frac{p}{1 + \varepsilon \cos\varphi} \qquad \frac{l\sqrt{2}}{\gamma\sqrt{m}} = p \qquad \vartheta - \vartheta_0 + \frac{\pi}{2} = \varphi$$

$\varepsilon = 0$ ($E_0 = V_0$) krožnica

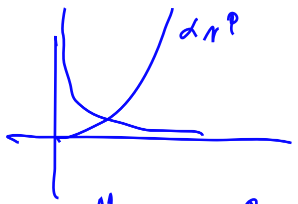
$\varepsilon = 1$ ($E_0 = 0$) parabola

$0 < \varepsilon < 1$ ($V_0 < E_0 < 0$) elipsa

$\varepsilon > 1$ ($E_0 > 0$) hiperbola

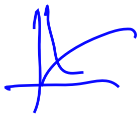
$$V = \alpha r^p \qquad V = \frac{l^2}{2 m r^2} + \alpha r^p$$

$\alpha > 0$; $p > 1$



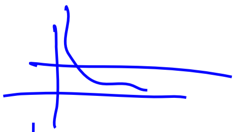
vsa gibanja so omejena; točka ne pade v center sile

$\alpha > 0$; $0 < p$



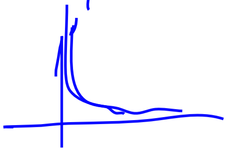
r^p; $0 < p < 1$

$\alpha > 0$, $p = 0$



ne pade v center sile, uleti v ∞

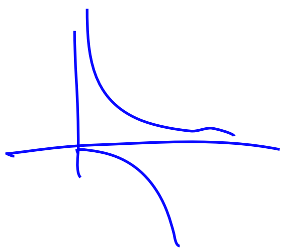
$\alpha > 0$, $p < 0$



ne pade v center sile, uleti v ∞

$\alpha < 0$, $p > 1$

$p \geq 0$



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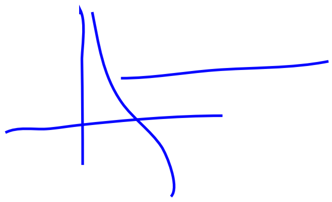
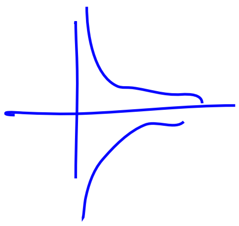
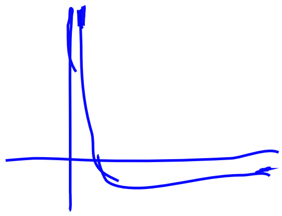


ne pade v center sile, uleti v ∞

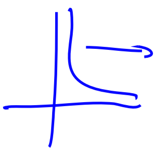
$\alpha < 0$, $p < 0$

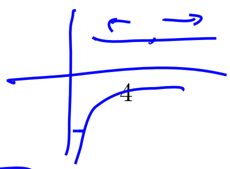


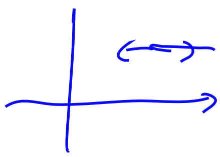
$-2 < p < 0$ ne pade v center sile; obstajajo omejene tirnice



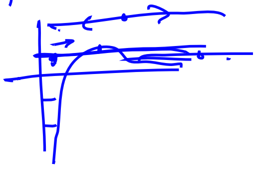
$\alpha < 0$, $p = -2$

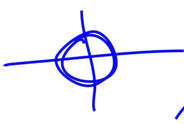






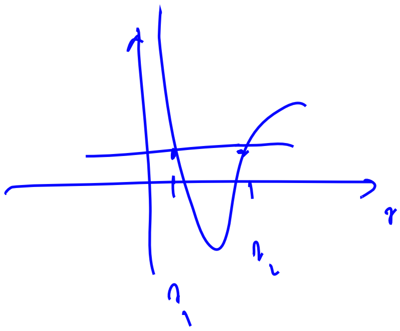
$\alpha < 0$, $p < -2$

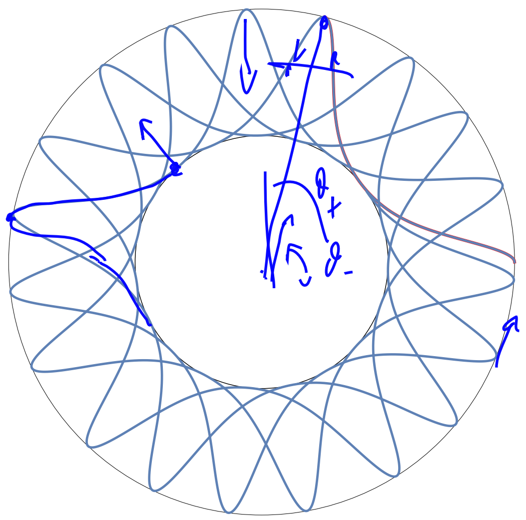




tir se spiralno navija na krožnico

Kvalitativna analiza gibanja v polju centralne sile





r_1, r_2 apsidna radija

$\dot r = 0$ pri $r = r_1, r_2$

Slika 1: Gibanje v polju centralne sile.

Apsidni radij, pericenter (r_1), apocenter (r_2), apsidni kot $\Delta\theta$.

Trditev 1. *Tir se dotika apsidnih krožnic.* ✓

$$\vec r\,' = \frac{d}{d\vartheta}(r\vec e_r) = \underbrace{\frac{\dot r}{\dot\vartheta}}_{=0}\vec e_r + r\vec e_\vartheta$$

Trditev 2. *Tir je simetričen glede na apsidni radij.*

$$\vartheta - \vartheta_0 = \operatorname{sgn}\dot r\,\frac{\ell}{\sqrt{2m}}\int_{r_0}^{r}\frac{dr}{r^2\sqrt{E_0 - U}}$$

$$\vartheta_- - \vartheta_0 = \frac{\ell}{\sqrt{2m}}\int_{r_0}^{r}$$

$$\vartheta_+ - \vartheta_0 = -\frac{\ell}{\sqrt{2m}}\int_{r_0}^{r}$$

Pogoj zaprtosti tira $W = \Delta\theta/\pi \in \mathbb{Q}$.

$$\vartheta_- - \vartheta_0 = -\vartheta_+ + \vartheta_0$$

$$\vartheta_0 - \vartheta_- = \vartheta_+ - \vartheta_0$$

Ali je tir zaprt?

$$\Delta\vartheta = \frac{\ell}{\sqrt{2m}}\int_{r_1}^{r_2}\frac{dr}{r^2\sqrt{E_0 - U(r)}}$$

$n\Delta\vartheta = 2\pi$ | $n\Delta\vartheta = m\,2\pi$

Tir je zaprt $\iff$ $W = \frac{\Delta\vartheta}{\pi} \in \mathbb{Q}$

Č omejeni tir ni zaprt je gosta množica v kolobarju med apsidnima radijema.

Bertrandov izrek: edina potenciala za katera je vsak omejen tir v okolici krožnega tira zaprt sta gravitacijski in Hookov.