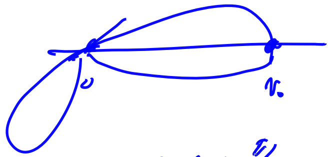
Vaje 24. november 2020



1. Točka se giblje v polju centralne sile po tiru $r = r_0 \cos n\varphi$, $n \in \mathbb{N}$.

$n\varphi = \pi/2$

   (a) Določi silo in pripadajoči efektivni potencial.
   (b) Za dani efektivni potencial kvalitativno obravnavaj možna gibanja.
   (c) Izračunaj energijo danega tira.
   (d) Določi čas, da iz točke največje oddaljenosti pade v center sile.

$1-\cos^2 n\varphi$

$a_r = -C_0^2 u^2 (u + u'')$ $\qquad u = \frac{1}{r} = \frac{1}{r_0}\cos^{-1} n\varphi$

$u' = +\frac{n}{r_0}\cos^{-2} n\varphi \sin n\varphi$ $\qquad u'' = \frac{n^2}{r_0}\left(+2\cos^{-3} n\varphi \sin^2 n\varphi + \cos^{-1} n\varphi\right)$

$a_r = -C_0^2 \frac{1}{r^2}\left(\frac{1}{r} + \frac{n^2}{r_0}\left(2\cos^{-3} n\varphi - 2\cos^{-1} n\varphi + \cos^{-1} n\varphi\right)\right)$

$2\frac{r_0^3}{r^3} - \frac{r_0}{r}$

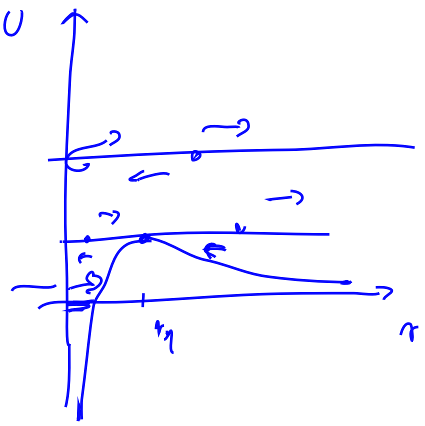
$\cos^{-1} n\varphi = \frac{r_0}{r}$

$a_r = -C_0^2 \frac{1}{r^2}\left(\frac{1}{r} + 2\frac{n^2 r_0^2}{r^3} - \frac{n^2}{r}\right) = -C_0^2\left((1-n^2)\frac{1}{r^3} + 2\frac{n^2 r_0^2}{r^5}\right)$

$\vec{F} = m a_r \vec{e}_r \Rightarrow F = -m C_0^2\left((1-n^2)\frac{1}{r^3} + 2\frac{n^2 r_0^2}{r^5}\right)$

$U = \frac{l^2}{2mr^2} + V$; $\quad V = -\underbrace{m C_0^2}_{\frac{l^2}{m}}\left((1-n^2)\frac{1}{2r^2} + \frac{n^2 r_0^2}{2r^4}\right)$; $\quad l = m C_0$

$U = \frac{l^2}{2m}\left(\frac{1}{r^2} - \frac{1-n^2}{r^2} - \frac{n^2 r_0^2}{r^4}\right) = \frac{l^2 n^2}{2m}\left(\frac{1}{r^2} - \frac{r_0^2}{r^4}\right)$

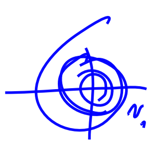


$U' = \frac{n^2 l^2}{2m}\left(-\frac{2}{r^3} + \frac{4 r_0^2}{r^5}\right)$

$4 r_0^2 = 2 r^2 \Rightarrow r_1 = \sqrt{2}\, r_0$

$U_1 = U(r_1) = \frac{l^2 n^2}{2m}\left(\frac{1}{2r_0^2} - \frac{r_0^2}{4 r_0^4}\right) = \frac{l^2 n^2}{8 m r_0^2}$

$E_0 > U_1$; $\quad E_0 = U_1$; $\quad E_0 < U_1$



$E_0 = T + V = \frac{1}{2} m \dot{r}^2 + U$ $\qquad r = r_0 \cos n\varphi$ $\qquad C_0 = r^2 \dot{\varphi}$

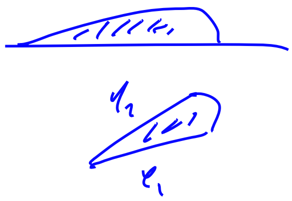
$$\dot r = \frac{dr}{dt} = \frac{dr}{d\varphi}\dot\varphi = (r_0\cos n\varphi)'\dot\varphi = -n r_0 \sin n\varphi\,\dot\varphi$$

$$\dot r^2 = n^2 r_0^2 \sin^2 n\varphi\,(\dot\varphi)^2 = n^2 r_0^2\left(1-\cos^2 n\varphi\right)\frac{C_0^2}{r^4} = n^2 r_0^2 C_0^2\left(\frac{1}{r^4}-\frac{1}{r_0^2 r^2}\right)$$

$$E_0 = \frac{1}{2}\frac{\ell^2}{m}\left(\frac{n^2 r_0^2}{r^4}-\frac{n^2}{r^2}\right) + \frac{n^2\ell^2}{2m}\left(\frac{1}{r^2}-\frac{r_0^2}{r^4}\right) = \frac{n^2\ell^2}{2m}(0) = 0$$

$$\frac{1}{2}C_0\cdot T = A = \frac{1}{2}\int_0^{\frac{\pi}{2n}} r_0^2\cos^2 n\varphi\,d\varphi \Rightarrow T = \frac{1}{C_0}r_0^2\frac{1}{n}\underbrace{\int_0^{\pi/2}\cos^2 z\,dz}_{=\int_0^{\pi/2}\sin^2 z\,dz} = \frac{r_0^2\pi}{4nC_0}$$

$$n\varphi = z$$



$$\frac{1}{2}\int_{\varphi_1}^{\varphi_2} r^2\,d\varphi$$

2. Točka se giblje pod vplivom centralne sile $\vec{F} = m\gamma r^{-3}\vec{e}_r$.

   (a) Kvalitativno obravnavaj možne tire.

   ~~(b) Zapiši energijo.~~

   (c) Izračunaj energijo danega tira.

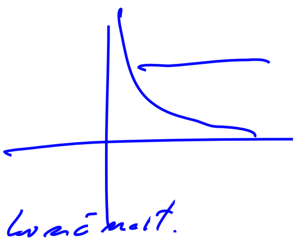
   $\to$ (d) Določi tir pri začetnem pogoju $r(t=0) = r_0$, $\dot{r}(t=0) = 0$ in $v(t=0) = v_0$.

$$F = m\gamma r^{-3} \quad ; \quad V = \frac{1}{2} m\gamma r^{-2} \qquad F = -\frac{dV}{dr}$$

$$V = \frac{\ell^2}{2mr^2} + \frac{m\gamma}{2r^2} = \frac{1}{2r^2}\left(\frac{\ell^2}{m} + m\gamma\right) \qquad \ell = m c_0$$

$$V = \frac{m}{2r^2}\left(c_0^2 + \gamma\right)$$

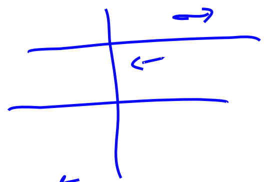
$c_0^2+\gamma>0$



točka vedno ubeži v neskončnost.

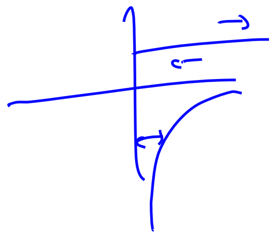
$E_0 > 0$

$c_0^2+\gamma=0$



$E_0 > 0$

$c_0^2+\gamma<0$



$E_0 > 0$

$E_0 < 0$ omejeno

$$\vartheta - \vartheta_0 = \operatorname{sgn}\dot{r}\, \frac{\ell}{\sqrt{2}} \int \frac{dr}{r^2\sqrt{E_0 - V(r)}}$$

$$\gamma u^3 = -c_0^2 u^2 (u + u'') \Rightarrow c_0^2 u'' + (c_0^2+\gamma)u = 0$$

$$u'' + \frac{c_0^2+\gamma}{c_0^2} u = 0$$

① $c_0^2+\gamma>0 \;;\quad \omega^2 = \frac{c_0^2+\gamma}{c_0^2} \;;\quad u'' + \omega^2 u = 0 \leftarrow A_1\cos\omega t + A_2 \sin\omega t$

$$\frac{1}{r} = u = A\cos(\omega\varphi - \delta) \Rightarrow r = \frac{1}{A\cos(\omega\varphi-\delta)}$$

② $c_0^2+\gamma = 0 \Rightarrow u'' = 0 \Rightarrow u' = A \Rightarrow u = A\varphi + B$

$$r = \frac{1}{A\varphi + B} \qquad \varphi = -\frac{B}{A}$$

③ $c_0^2+\gamma<0 \qquad \omega^2 = -\frac{c_0^2+\gamma}{c_0^2} \;;\quad u'' - \omega^2 u = 0$

$$u = A\,\mathrm{ch}\,\omega\varphi + B\,\mathrm{sh}\,\omega\varphi = C\,\mathrm{sh}(\omega\varphi + \delta) =$$
$$= C(\mathrm{sh}\,\omega\varphi\,\mathrm{ch}\,\delta + \mathrm{sh}\,\delta\,\mathrm{ch}\,\omega\varphi)$$

$A = C\,\mathrm{sh}\,\delta$
$B = C\,\mathrm{ch}\,\delta$ $\quad B^2 - A^2 = c^2(\mathrm{ch}^2\delta - \mathrm{sh}^2\delta) = \underline{c^2}$ $\quad \underline{B^2 > A^2}$

$$r = \frac{1}{A\,\mathrm{ch}\,\omega\varphi + B\,\mathrm{sh}\,\omega\varphi}$$

$r(t=0) = r_0$; $\dot{r}(t=0) = 0$; $v(t=0) = v_0$; $\varphi(t=0) = 0$

(2) $\varphi(t=0)=0 \Rightarrow$

$$r = \frac{1}{A\varphi + B} = r_0 = \frac{1}{B}$$

$$\dot{r} = \frac{-1\,A}{(A\varphi+B)^2}\dot{\varphi} \Rightarrow \dot{\varphi} = 0 \text{ al. } \underline{A = 0}$$

$C_0 = r^2\dot{\varphi}$

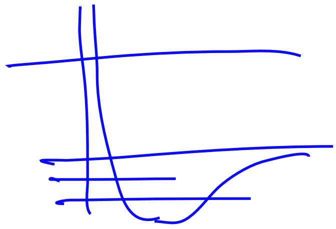
$\dot{\varphi} = 0 \Rightarrow C_0 = 0 \Rightarrow u = 0$; $u = \frac{1}{r}$

$r = r_0$ $\quad$ $\underline{\text{krożnice}}$ $\quad v_0 = r_0\dot{\varphi} \Rightarrow \dot{\varphi} = \frac{v_0}{r_0}$

$C_0 = r_0^2\dot{\varphi} = \underline{r_0 v_0}$

$$\underline{\gamma = -C_0^2 = -r_0^2 v_0^2}$$

3. Točka se giblje pod vplivom potenciala $V = -\alpha r^{-1} + \beta r^{-2}$, $\alpha, \beta > 0$.



(a) Kvalitativno obravnavaj možne tire.

(b) Določi tir.

$$V = -\frac{\gamma}{r} \qquad V = \frac{l^2}{2m r^2} - \frac{\alpha}{r} + \frac{\beta}{r^2} = \left(\frac{l^2}{2m} + \beta\right)\frac{1}{r^2} - \frac{\alpha}{r} \quad \Big| \quad U(r_1) = E_0$$

$$\vartheta - \vartheta_1 = \frac{l}{\sqrt{2m}} \int_{r_1}^{r} \frac{dr}{r^2 \sqrt{E_0 - \left(\frac{l^2}{2m} + \beta\right)\frac{1}{r^2} + \frac{\alpha}{r}}} \qquad u = \frac{1}{r}\ ;\ du = -\frac{1}{r^2}\,dr$$

$$\vartheta - \vartheta_1 = -\frac{l}{\sqrt{2m}} \int_{u_1}^{u} \frac{du}{\sqrt{E_0 - \left(\frac{l^2}{2m} + \beta\right)u^2 + \alpha u}} =$$

$$= -\frac{l}{\sqrt{2m}}\,\frac{1}{\sqrt{\frac{l^2}{2m} + \beta}} \int_{u_1}^{u} \frac{du}{\sqrt{\frac{E_0}{\frac{l^2}{2m}+\beta} - u^2 + \frac{\alpha}{\frac{l^2}{2m}+\beta}\,u}}$$

$$\gamma = \frac{l}{\sqrt{2m}}\,\frac{1}{\sqrt{\frac{l^2}{2m} + \beta}}$$

$$\beta = 0 \qquad \frac{l}{\sqrt{2m}}\,\frac{1}{\sqrt{\frac{l^2}{2m} + \beta}} = 1$$

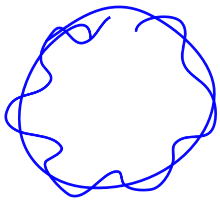
$$\vartheta - \vartheta_1 = \gamma \arccos\left(\frac{u - a}{b}\right) - \vartheta_0$$

$$\cos \frac{1}{\gamma}(\vartheta - \vartheta_1 + \vartheta_0) = \frac{u - a}{b} \Rightarrow u = a + b \cos \frac{1}{\gamma}(\vartheta - \vartheta_1 + \vartheta_0)$$

$$r = \frac{1}{a + b \cos \frac{1}{\gamma}(\vartheta - \vartheta_1 + \vartheta_0)}$$

$$r = \frac{p}{1 + \varepsilon \cos\left(\frac{1}{\gamma}\right)(\vartheta - \vartheta_1 + \vartheta_0)}$$

ni stožnica za $\gamma \neq 1$

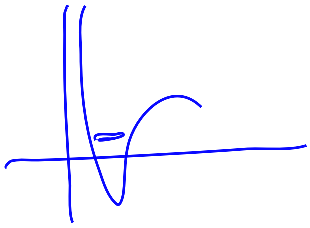


$\frac{1}{2} \in Q$

$\frac{1}{2} \notin Q$ fi- ri
zgomot

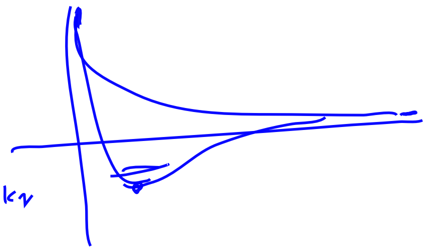
4. Točka se giblje v polju centralne sile pod vplivom potenciala $V = -\frac{\alpha}{r}\,\mathrm{e}^{-kr}$. Določi vrtilno količino tako, da bo gibanje omejeno.

$$U = \frac{l^2}{2mr^2} + V$$



$$\alpha > 0,\quad k > 0$$

$$U = \frac{l^2}{2mr^2} - \frac{\alpha}{r} e^{-kr}$$



$$U' = -\frac{l^2}{mr^3} + \frac{\alpha}{r^2} e^{-kr} + \frac{\alpha k}{r} e^{-kr}$$

$$\frac{l^2}{m} = \underbrace{\alpha r e^{-kr} + \alpha k r^2 e^{-kr}}$$ ta enačba mora biti rešljiva

$$f(r) = \alpha r e^{-kr} + \alpha k r^2 e^{-kr} \qquad \max f(r) \geq \frac{l^2}{m}$$

$$f' = \alpha\left(e^{-kr} - kr e^{-kr} + 2kr e^{-kr} - \cancel{2}k^2 r^2 e^{-kr}\right) =$$

napaka

$$= \alpha e^{-kr}\left(1 + kr - \cancel{2}k^2 r^2\right)$$

$$r_{1,2} = \frac{-k \pm \sqrt{k^2 + 4k^2}}{-k^2} = \frac{1 \pm \sqrt{5}}{k}$$

$$1 + kr - 2k^2r^2 = 0 \Rightarrow r_{1,2} = \frac{-k \pm \sqrt{k^2 + 8k^2}}{-4k^2} = \frac{-1 \pm 3}{-4k}$$

$$r_1 = \frac{1+3}{4k} = \frac{1}{k} \qquad r_1 = \frac{1+\sqrt{5}}{k}$$

$$f\left(\frac{1}{k}\right) = \alpha \frac{1}{k} e^{-1} + \alpha \frac{1}{k} e^{-1} = \frac{2\alpha}{ke} \geq \frac{l^2}{m} \Rightarrow l^2 \leq \frac{2\alpha m}{ke}$$

$$f\left(\frac{1+\sqrt{5}}{k}\right) = \ldots$$

$$l < \sqrt{\frac{2\alpha m}{ke}}$$