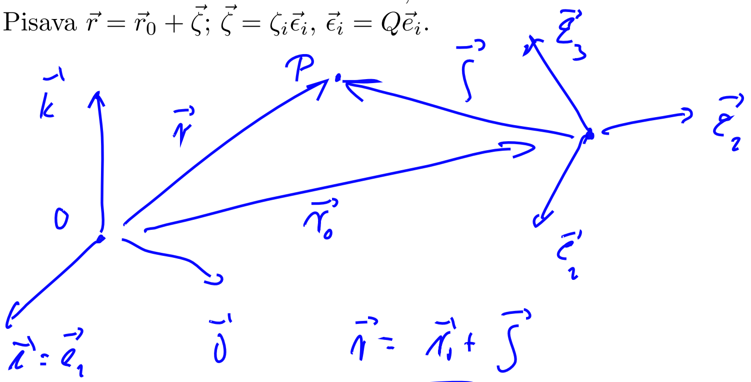
Predavanje 30. november 2020

Relativno gibanje

Absolutni koordinatni sistem AKS, relativni koordinatni sistem RKS.
Pisava $\vec{r} = \vec{r}_0 + \vec{\zeta}$; $\vec{\zeta} = \zeta_i \vec{\epsilon}_i$, $\vec{\epsilon}_i = Q\vec{e}_i$.



$\underline{\vec{r} = \vec{r}_1 + \vec{\zeta}}$

$\vec{\varepsilon}_i$ za AKS niso konstantni.
So funkcija časa
$\vec{e}_i$ za RKS so konstantni

$$\vec{\zeta} = \sum_{i=1}^{3} \zeta_i \vec{\varepsilon}_i = \zeta_i \vec{\varepsilon}_i$$

sumacijska konvencija
dogovor

$$\vec{\varepsilon}_i = Q\vec{e}_i \quad ; \quad Q = Q(t)$$

↑ rotacija

Izrek 1. *Obstaja vektor $\vec{\omega}$ tako, da je $\dot{\vec{\epsilon}}_i = \vec{\omega} \times \vec{\epsilon}_i$. Vektorju $\vec{\omega}$ pravimo vektor kotne hitrosti.*

$$\dot{\vec{\varepsilon}}_i = \vec{\omega} \times \vec{\varepsilon}_i \quad ; \quad \vec{\omega} = \vec{\omega}(t)$$

$$\dot{\vec{\varepsilon}}_1 = \cancel{a_{11}}\vec{\varepsilon}_1 + a_{12}\vec{\varepsilon}_2 + a_{13}\vec{\varepsilon}_3 \quad / \cdot \vec{\varepsilon}_2 \quad / \vec{\varepsilon}_1 \qquad \vec{\varepsilon}_1 \cdot \vec{\varepsilon}_1 = 1 \Rightarrow \dot{\vec{\varepsilon}}_1 \cdot \vec{\varepsilon}_1 = 0$$

$$\dot{\vec{\varepsilon}}_2 = a_{21}\vec{\varepsilon}_1 + \cancel{a_{22}}\vec{\varepsilon}_2 + a_{23}\vec{\varepsilon}_3 \quad / \cdot \vec{\varepsilon}_1 \qquad \dot{\vec{\varepsilon}}_1 \perp \vec{\varepsilon}_1 \Rightarrow a_{11} = 0$$

$$\dot{\vec{\varepsilon}}_3 = a_{31}\vec{\varepsilon}_1 + a_{32}\vec{\varepsilon}_2 + \cancel{a_{33}}\vec{\varepsilon}_3 \qquad a_{22} = a_{33} = 0$$

$$\vec{\varepsilon}_1 \cdot \vec{\varepsilon}_2 = 0 \Rightarrow \dot{\vec{\varepsilon}}_1 \cdot \vec{\varepsilon}_2 + \vec{\varepsilon}_1 \cdot \dot{\vec{\varepsilon}}_2 = 0$$

$$a_{12} + a_{21} = 0 \Rightarrow a_{12} = -a_{21}$$

$$\dot{\vec{\varepsilon}}_i = A \begin{bmatrix} \vec{\varepsilon}_1 \\ \vec{\varepsilon}_2 \\ \vec{\varepsilon}_3 \end{bmatrix} \qquad A^T = -A \qquad a_{13} = -a_{31}, \; a_{23} = -a_{32}$$

A je poševno simetrična matrika

$$a_{12}, \; a_{13}, \; a_{23} \qquad \vec{\omega} = \omega_i \vec{\varepsilon}_i$$

$$\dot{\vec{\varepsilon}}_1 = \vec{\omega} \times \vec{\varepsilon}_1 = (\omega_1\vec{\varepsilon}_1 + \omega_2\vec{\varepsilon}_2 + \omega_3\vec{\varepsilon}_3) \times \vec{\varepsilon}_1 = -\omega_2\vec{\varepsilon}_3 + \omega_3\vec{\varepsilon}_2 \Rightarrow \boxed{a_{13} = -\omega_2} \quad \boxed{a_{12} = \omega_3}$$

$$\dot{\vec{\varepsilon}}_2 = \vec{\omega} \times \vec{\varepsilon}_2 = (\omega_1\vec{\varepsilon}_1 + \omega_2\vec{\varepsilon}_2 + \omega_3\vec{\varepsilon}_3) \times \vec{\varepsilon}_2 = \omega_1\vec{\varepsilon}_3 - \omega_3\vec{\varepsilon}_1$$

$$\Rightarrow \omega_1 = a_{23}, \; \boxed{-\omega_3 = a_{21}} \; \checkmark$$

$\dot{\vec\varepsilon}_3 = \vec\omega \times \vec\varepsilon_3 = (\omega_1\vec\varepsilon_1 + \omega_2\vec\varepsilon_2 + \omega_3\vec\varepsilon_3)\times\vec\varepsilon_3 = -\omega_1\vec\varepsilon_2 + \omega_2\vec\varepsilon_1$

$\Rightarrow -\omega_1 = a_{32}, \quad \omega_2 = a_{31}$

$Q = R(\vec e, \varphi)$ $\quad \dot{\vec\varepsilon}_i = \vec\omega\times\vec\varepsilon_i$ $\quad Q\vec e = \vec e$

Os rotacije je lastni vektor rotacije

Trditev 1. *Vektor kotne hitrosti rotacije $R(\vec e, \varphi)$ okrog stalne osi je $\vec\omega = \dot\varphi\vec e$.*

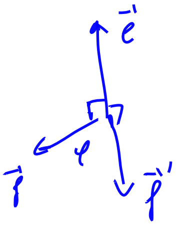
$\vec e = e_i\vec e_i$

$Q\vec e = e_i Q\vec e_i = e_i\vec\varepsilon_i$ $\Big\} \Rightarrow \boxed{e_i\vec e_i} = e_i\vec\varepsilon_i = \vec e$

- $\dot{\vec e} = \vec 0 \Rightarrow \vec 0 = \dot e_i\vec\varepsilon_i + e_i\dot{\vec\varepsilon}_i$
- $\vec e = e_i\vec e_i \Rightarrow \vec 0 = \dot e_i\vec e_i + e_i\dot{\vec e}_i \Rightarrow \dot e_i = 0$

$\vec 0 = e_i\dot{\vec\varepsilon}_i = e_i\vec\omega\times\vec\varepsilon_i = \vec\omega\times e_i\vec\varepsilon_i = \vec\omega\times\vec e$

$\boxed{\vec 0 = \vec\omega\times\vec e} \Rightarrow \vec\omega \parallel \vec e \qquad \vec\omega = \omega\vec e \qquad \omega = \dot\varphi$



$\cos\varphi = \vec f\cdot\vec f' = \vec f\cdot Q\vec f$

$\vec f = f_i\vec e_i \; ; \; Q\vec f = f_i Q\vec e_i = f_i\vec\varepsilon_i$

$\cos\varphi = \vec f\cdot f_i\vec\varepsilon_i$

$-\sin\varphi\cdot\dot\varphi = \vec f\cdot f_i\dot{\vec\varepsilon}_i = \vec f\cdot f_i\vec\omega\times\vec\varepsilon_i = \vec f\cdot(\vec\omega\times Q\vec f)$

$-\sin\varphi\,\dot\varphi = \vec\omega\cdot(Q\vec f\times\vec f) = \vec\omega\cdot(\vec f'\times\vec f) = (\omega\vec e)\cdot(-\vec e\sin\varphi)$

$-\sin\varphi\,\dot\varphi = -\omega\sin\varphi \Rightarrow \boxed{\omega = \dot\varphi} \qquad -\vec e|\vec f||\vec f'|\sin\varphi$

$\vec e = \vec e(t), \quad \varphi = \varphi(t)$

$\vec\omega = \dot\varphi\vec e + \sin\varphi\,\dot{\vec e} + (1-\cos\varphi)\vec e\times\dot{\vec e}$

Q ; $\det Q = 1$; $QQ^T = I \Rightarrow Q$ je rotacija

$Q\vec e = \vec e$; os je lastni vektor

$\cos\varphi = \frac{1}{2}(\mathrm{sl}\,Q - 1)$

Zveza med relativno hitrostjo in absolutno hitrostjo

$\vec{r}\,; \quad \dot{\vec{r}} = \vec{v}$

$\vec{\zeta} = \underbrace{\zeta_i \vec{\epsilon}_i} \Rightarrow \dot{\vec{\zeta}} = \underline{\dot{\zeta}_i \vec{\epsilon}_i} + \zeta_i \dot{\vec{\epsilon}}_i$

Relativni opazovalec vidi samo $\underbrace{\dot{\zeta}_i \vec{\epsilon}_i}$

$\underline{\dot{\zeta}_i \vec{\epsilon}_i} = \vec{v}_{rel}$ vektor relativne hitrosti

Relativna hitrost $\vec{v}_{\text{rel}} = \dot{\zeta}_i \vec{\epsilon}_i$.

$\zeta_i \vec{\omega} \times \vec{\epsilon}_i = \vec{\omega} \times \vec{\zeta}$

$\vec{r} = \vec{r}_0 + \vec{\zeta}$

$\vec{v} = \dot{\vec{r}} = \dot{\vec{r}}_0 + \dot{\vec{\zeta}} = \vec{v}_0 + \dot{\zeta}_i \vec{\epsilon}_i + \zeta_i \dot{\vec{\epsilon}}_i = \vec{v}_0 + \vec{\omega} \times \vec{\zeta} + \vec{v}_{rel}$

hitrost izhodišča RKS

translatorna hitrost

rotacijska hitrost

Zveza
$$\vec{v} = \vec{v}_0 + \vec{\omega} \times \vec{\zeta} + \vec{v}_{\text{rel}}.$$

$\vec{a}_{rel} = \ddot{\zeta}_i \vec{\epsilon}_i$

$\vec{a} = \dot{\vec{v}} = \dot{\vec{v}}_0 + \dot{\vec{\omega}} \times \vec{\zeta} + \vec{\omega} \times \dot{\vec{\zeta}} + \dot{\vec{v}}_{rel} =$

$= \dot{\vec{v}}_0 + \dot{\vec{\omega}} \times \vec{\zeta} + \vec{\omega} \times (\vec{\omega} \times \vec{\zeta} + \vec{v}_{rel}) + \frac{d}{dt}(\dot{\zeta}_i \vec{\epsilon}_i)$

$\dot{\vec{v}}_0 = \vec{a}_0$

$\frac{d}{dt}(\dot{\zeta}_i \vec{\epsilon}_i) = \ddot{\zeta}_i \vec{\epsilon}_i + \dot{\zeta}_i \dot{\vec{\epsilon}}_i = \vec{a}_{rel} + \dot{\zeta}_i \vec{\omega} \times \vec{\epsilon}_i$

$= \vec{a}_{rel} + \vec{\omega} \times \vec{v}_{rel}$

$$\boxed{\vec{a} = \vec{a}_0 + \dot{\vec{\omega}} \times \vec{\zeta} + \vec{\omega} \times (\vec{\omega} \times \vec{\zeta}) + 2\vec{\omega} \times \vec{v}_{rel} + \vec{a}_{rel}}$$

Zveza med relativnim pospeškom in absolutnim pospeškom

Relativni pospešek $\vec{a}_{\mathrm{rel}} = \ddot{\zeta}_i \vec{\epsilon}_i$.

Zveza

$$\vec{a} = \overbrace{\vec{a}_0 + \vec{\omega} \times (\vec{\omega} \times \vec{\zeta}) + \dot{\vec{\omega}} \times \vec{\zeta}}^{\text{sistemski pospešek}} + 2\vec{\omega} \times \vec{v}_{\mathrm{rel}} + \vec{a}_{\mathrm{rel}}.$$

Sistemski, Coriolisov, relativni pospešek.

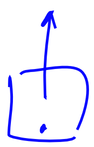
$2\vec{\omega} \times \vec{v}_{rel}$ $\qquad$ $\vec{a}_{rel}$

$$\vec{a}_s = \vec{a}_0 + \vec{\omega} \times (\vec{\omega} \times \vec{\zeta}) + \dot{\vec{\omega}} \times \vec{\zeta}$$

centripetalni pospešek $\qquad$ Eulerjev pospešek

Posebni primer: $\dot{\vec{\omega}} = \vec{0}$; $\vec{a}_0 = \vec{0}$; $\vec{v}_{rel} = \vec{0}$, $\vec{a}_{rel} = \vec{0}$

$$\vec{a} = \vec{\omega} \times (\vec{\omega} \times \vec{\zeta})$$



$\vec{a} = \vec{a}_0$

Newtonova enačba v relativnem koordinatnem sistemu

AKS inercialen: $m\vec{a} = \vec{F}$

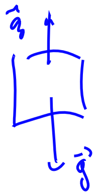
$$m\left(\vec{a}_0 + \dot{\vec{\omega}}\times\vec{\zeta} + \vec{\omega}\times(\vec{\omega}\times\vec{\zeta}) + 2\vec{\omega}\times\vec{v}_{rel} + \vec{a}_{rel}\right) = \vec{F}$$

$$m\vec{a}_{rel} = \vec{F} - m\vec{a}_0 - m\dot{\vec{\omega}}\times\vec{\zeta} - m\vec{\omega}\times(\vec{\omega}\times\vec{\zeta}) - 2m\vec{\omega}\times\vec{v}_{rel} = \vec{F}_{rel}$$

$$m\vec{a}_{rel} = \vec{F}_{rel}$$

fiktivne sile

$\vec{0} = \vec{S} + m\vec{g} - m\vec{a}_0 = F_{rel}$



$\vec{S} = m(\vec{a}_0 - \vec{g}) = m(g + a_0)\vec{k}$

$\vec{g} = -g\vec{k}$; $\vec{a}_0 = a\vec{k}$

$-2m\vec{\omega}\times\vec{v}_{rel}$ Coriolisova sila

$-m\vec{\omega}\times(\vec{\omega}\times\vec{\zeta})$ centrifugalna sila

$-m\dot{\vec{\omega}}\times\vec{\zeta}$ Eulerjeva sila

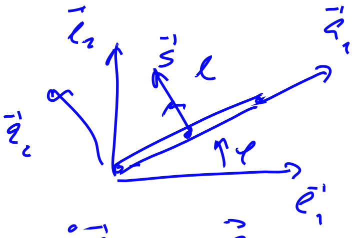
$-m\vec{a}_0$ inercijska sila

$$m\vec{a}_{\mathrm{rel}} = \vec{F} - m\vec{a}_0 - m\vec{\omega}\times(\vec{\omega}\times\vec{\zeta}) - m\dot{\vec{\omega}}\times\vec{\zeta} - 2m\vec{\omega}\times\vec{v}_{\mathrm{rel}}.$$

Translacijska sila, centrifugalna sila, inercijska sila rotacije, Coriolisova sila.

Primer

Gibanje točke v vrteči se cevi.   .



$t = 0$

Kdaj se izpusti se

$\vec{S} = S\vec{\varepsilon}_2$

gibanje brez trenja

$$\vec{\omega} = \dot{\varphi}\vec{e}_3 = \dot{\varphi}\vec{\varepsilon}_3 = \omega_0\vec{\varepsilon}_3$$

$$m\vec{a}_{rel} = \vec{S} - \underset{0}{m\vec{a}_0} - m\underset{0}{\dot{\vec{\omega}}}\times\vec{\zeta} - m\vec{\omega}\times(\vec{\omega}\times\vec{\zeta}) - 2m\vec{\omega}\times\vec{v}_{rel}$$

$$\vec{\omega}\times\vec{\zeta} = \vec{\omega}\times\zeta_1\vec{\varepsilon}_1 = \omega_0\zeta_1\vec{\varepsilon}_3\times\vec{\varepsilon}_1 = \omega_0\zeta_1\vec{\varepsilon}_2$$

$$\vec{\omega}\times(\vec{\omega}\times\vec{\zeta}) = \omega_0^2\zeta_1\vec{\varepsilon}_3\times\vec{\varepsilon}_2 = -\omega_0^2\zeta_1\vec{\varepsilon}_1$$

$$\vec{\omega}\times\vec{v}_{rel} = \omega_0\vec{\varepsilon}_3\times\dot{\zeta}_1\vec{\varepsilon}_1 = \omega_0\dot{\zeta}_1\vec{\varepsilon}_2$$

$$m\ddot{\zeta}_1\vec{\varepsilon}_1 = S\vec{\varepsilon}_2 + m\omega_0^2\zeta_1\vec{\varepsilon}_1 - 2m\omega_0\dot{\zeta}_1\vec{\varepsilon}_2$$

$$m\ddot{\zeta}_1 = m\omega_0^2\zeta_1 \quad \longleftarrow$$

$$0 = S - 2m\omega_0\dot{\zeta}_1$$

$$\ddot{\zeta}_1 - \omega_0^2\zeta_1 = 0$$

$$\zeta_1 = A\operatorname{sh}\omega t + B\operatorname{ch}\omega t$$

$$\xi_1(t=0) = \frac{\ell}{2} \; ; \quad \dot{\xi}_1(t=0) = 0 \Rightarrow \omega A \operatorname{ch} \omega t + \omega B \operatorname{sh} \omega t = \dot{\xi}_1$$

$$\Rightarrow A = 0$$

$$\xi_1 = \frac{\ell}{2} \operatorname{ch} \omega_0 t$$

$$\frac{\ell}{2} \operatorname{ch} \omega_0 t = \ell \Rightarrow \operatorname{ch} \omega_0 t = 2$$

$$\boxed{t = \frac{1}{\omega_0} \operatorname{Arch} 2}$$

Foucaultovo nihalo