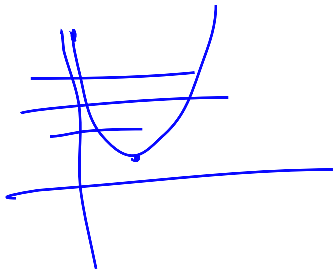
Vaje 1. december 2020

1. Točka se giblje pod vplivom centralne sile $\vec{F} = -kr\vec{e}_r$, $k > 0$.

   (a) Kvalitativno obravnavaj možne tire.

   (b) Določi tir.

$$\underbrace{V = \frac{1}{2}kr^2}$$



$$U = \frac{l^2}{2mr^2} + \frac{1}{2}kr^2$$

$$\rightarrow \quad \vartheta - \vartheta_0 = \operatorname{sgn}\dot{r}\,\frac{l}{\sqrt{2m}}\int_{r_0}^{r}\frac{dr}{r^2\sqrt{E-V}}$$

$$m\vec{a} = \vec{F} = -kr\vec{e}_r \qquad r\vec{e}_r = r(\cos\vartheta\,\vec{i} + \sin\vartheta\,\vec{j}) = x\vec{i} + y\vec{j}$$

$$m\ddot{x} = -kx \qquad \omega^2 = \frac{k}{m}$$

$$m\ddot{y} = -ky$$

$$x = A_1\cos(\omega t - \delta_1) \qquad y = A_2\cos(\omega t - \delta_2)$$

$$x = A_1\cos\omega t\cos\delta_1 + A_1\sin\omega t\sin\delta_1$$

$$y = A_2\cos\omega t\cos\delta_2 + A_2\sin\omega t\sin\delta_2$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \underbrace{\begin{bmatrix} A_1\cos\delta_1 & A_1\sin\delta_1 \\ A_2\cos\delta_2 & A_2\sin\delta_2 \end{bmatrix}}_{A}\begin{bmatrix} \cos\omega t \\ \sin\omega t \end{bmatrix}$$

$$\det A = A_1A_2(\cos\delta_1\sin\delta_2 - \cos\delta_2\sin\delta_1)$$

$$= A_1A_2\sin(\delta_2 - \delta_1)$$

$A_1A_2 \neq 0$ in $\delta_1 \neq \delta_2$

$A_1A_2 = 0 \Rightarrow A_1 = 0,\ A_2 \neq 0;\ A_1 \neq 0,\ A_2 = 0 \}$ premočrtno gibanje

$\delta_1 = \delta_2 \qquad \frac{y}{x} = \frac{A_2}{A_1}$

$$\begin{bmatrix} \cos\omega t \\ \sin\omega t \end{bmatrix} = A^{-1}\begin{bmatrix} x \\ y \end{bmatrix}$$

kvadratna forma

$$1 = \cos^2\omega t + \sin^2\omega t = \overbrace{\alpha x^2 + \beta xy + \gamma y^2 + \delta x + \zeta y + \varepsilon}^{\text{kvadratna forma}} = 1$$

Tiri so elipse
Center sile je v središču elipse

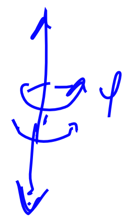
$$|\vec e| = 1$$

$\pm$

2. Zapis rotacije . $R(\vec e,\varphi)\vec r = \cos\varphi\,\vec r + (\vec r\cdot\vec e)(1-\cos\varphi)\vec e + \vec e\times\vec r\,\sin\varphi$

Formula velja za $\vec r \parallel \vec e$

$$R(\vec e,\varphi)\alpha\vec e = \cos\varphi\,\underbrace{\alpha\vec e}_{\vec r} + \underbrace{(\alpha)(1-\cos\varphi)\vec e}_{\vec r} + \vec 0 = \cos\varphi\,\vec r + (1-\cos\varphi)\vec r = \vec r \checkmark$$

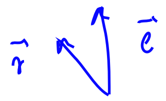


$\vec r \nparallel \vec e$

$\vec r, \vec e, \vec e\times\vec r$ so linearno neodvisni

$$\vec r' = R(\vec e,\varphi)\vec r = \alpha\vec r + \beta\vec e + \gamma\,\vec e\times\vec r$$

1. $\vec r\cdot\vec e = \vec r'\cdot\vec e$ $\qquad \vec r\cdot\vec e = \vec r'\cdot\vec e = \alpha\,\vec r\cdot\vec e + \beta \Rightarrow \beta = \vec r\cdot\vec e\,(1-\alpha)$
2. $|\vec r|^2 = |\vec r'|^2$ $\qquad |\vec r|^2 = \alpha^2|\vec r|^2 + \beta^2 + \gamma^2\underbrace{(\vec e\times\vec r)\cdot(\vec e\times\vec r)}_{|\vec r|^2-(\vec e\cdot\vec r)^2} + 2\alpha\beta\,\vec r\cdot\vec e \quad =$



3. $(\vec e\times\vec r)\cdot(\vec e\times\vec r') = |\vec e\times\vec r||\vec e\times\vec r'|\cos\varphi$

$$0 = |\vec r|^2(-1+\alpha^2+\gamma^2) + (\vec e\cdot\vec r)^2\left(-\gamma^2 + (1-\alpha)^2 + 2\alpha(1-\alpha)\right)$$

$$-\gamma^2 + 1 - 2\alpha + \alpha^2 + 2\alpha - 2\alpha^2 = 1-\alpha^2-\gamma^2$$

$$0 = |\vec r|^2(-1+\alpha^2+\gamma^2) + (\vec e\cdot\vec r)^2(1-\alpha^2-\gamma^2) \quad \text{za vsak } \vec r$$

$$1 = \alpha^2+\gamma^2 \Rightarrow \alpha = \cos\psi \quad \gamma = \sin\psi \qquad \psi = \varphi$$

$$|\vec e\times\vec r|^2\cos\varphi = (\vec e\times\vec r)\cdot(\alpha\,\vec e\times\vec r + \gamma\,\underbrace{\vec e\times(\vec e\times\vec r)}) =$$

$$= \alpha|\vec e\times\vec r|^2 \Rightarrow \alpha = \cos\varphi \qquad \cos\varphi = \cos\psi$$

$$\Rightarrow \varphi = \pm\psi + k2\pi$$

$Q\vec e = \vec e$ določa os rotacije

$\operatorname{Sl} Q = \vec e_i\cdot Q\vec e_i$

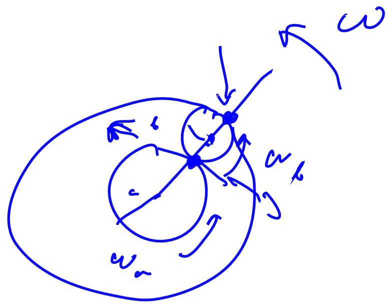
$\vec i = \vec e, \vec j, \vec k$

$$Q \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{bmatrix}$$

$$\operatorname{Sl} Q = 1 + 2\cos\varphi$$

$$\cos\varphi = \frac{\operatorname{Sl} Q - 1}{2}$$

3. Planetno gonilo. .



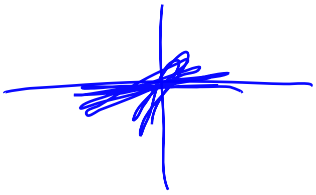
$$a\omega_a = (a+b)\omega - b\omega_b$$
$$0 = (a+b)\omega + b\omega_b$$

(1) ω predpisa $\Rightarrow$ $b\omega_b = -(a+b)\omega$

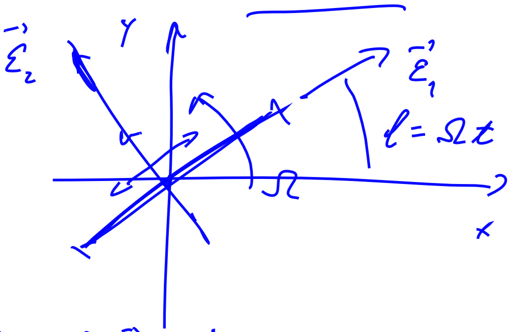
$$a\omega_a = 2(a+b)\omega \Rightarrow \omega_a = \frac{2(a+b)}{a}\omega$$

(2) ω_a predpisa

$$\omega_b = -\frac{(a+b)\omega}{b} = -\frac{(a+b)\cdot a}{b\cdot 2(a+b)}\omega_a = -\frac{a}{2b}\omega_a$$



4. Točka harmonično niha s frekvenco ω, smer nihanja pa enakomerno kroži okoli središčne lege s kotno hitrostjo Ω.
   (a) Izračunaj hitrost in pospešek.
   (b) Obravnavaj posebni primer $\omega = \Omega$. Določi tudi tir.



$\vec\rho = \rho_i \vec\varepsilon_i$; $\rho_1 = A\cos\omega t$

$\rho_2 = \rho_3 = 0$

$\vec\omega = \Omega \vec k = \Omega \vec\varepsilon_3$

$\vec r = A\cos\omega t\, \vec e_r$

$\vec v = \vec v_0 + \vec\omega\times\vec\rho + \vec v_{rel}$, $\quad \vec v_0 = 0$

$\vec\omega\times\vec\rho = \Omega\vec\varepsilon_3\times\rho_1\vec\varepsilon_1 = \Omega\rho_1\vec\varepsilon_2$; $\quad \vec v_{rel} = \dot\rho_1\vec\varepsilon_1$

$\vec v = \Omega A\cos\omega t\,\vec\varepsilon_2 - \omega A\sin\omega t\,\vec\varepsilon_1 \qquad \vec\varepsilon_1 = \vec e_r$; $\vec\varepsilon_2 = \vec e_\varphi$

$\vec a = \vec a_0 + \dot{\vec\omega}\times\vec\rho + \vec\omega\times(\vec\omega\times\vec\rho) + 2\vec\omega\times\vec v_{rel} + \vec a_{rel}$, $\quad \vec a_0 = 0$, $\quad \dot{\vec\omega} = \vec 0$

$\dot\Omega = 0$

$\vec\omega\times(\vec\omega\times\vec\rho) = \rho_1\Omega^2\vec\varepsilon_3\times\vec\varepsilon_2 = -\rho_1\Omega^2\vec\varepsilon_1$

$\vec\omega\times\vec v_{rel} = \Omega\dot\rho_1\vec\varepsilon_2 \qquad \vec a_{rel} = \ddot\rho_1\vec\varepsilon_1$

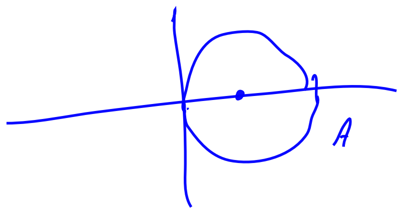
$\vec a = -A\cos\omega t\,\Omega^2\vec\varepsilon_1 + 2\Omega A(-\omega\sin\omega t)\vec\varepsilon_2 - A\omega^2\cos\omega t\,\vec\varepsilon_1$

$= A\left[-(\omega^2+\Omega^2)\cos\omega t\,\vec\varepsilon_1 - 2\omega\Omega\sin\omega t\,\vec\varepsilon_2\right]$

$\vec r = A\cos\omega t(\cos\omega t\,\vec i + \sin\omega t\,\vec j) =$

$= A(\cos^2\omega t\,\vec i + \frac{1}{2}\sin 2\omega t\,\vec j) =$

$= A(\frac{1}{2}(1+\cos 2\omega t)\vec i + \frac{1}{2}\sin 2\omega t\,\vec j) = \frac{1}{2}A\vec i + \frac{1}{2}A(\cos 2\omega t\,\vec i + \sin 2\omega t\,\vec j)$

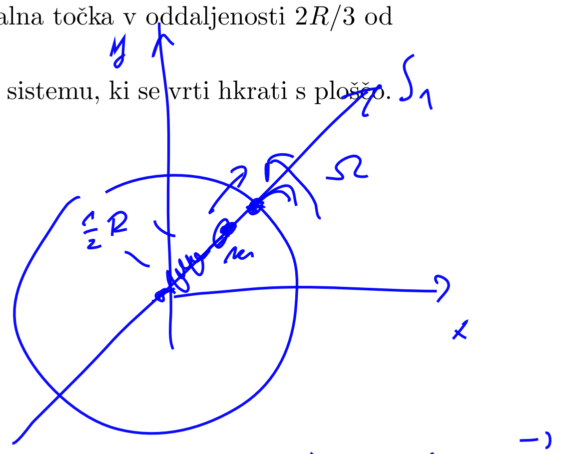


5. Vodoravno postavljena okrogla plošča s polmerom R se enakomerno s kotno hitrostjo Ω vrti okrog svoje osi. Na plošči je radialno vodilo, po kateri se brez trenja giblje materialna točka z maso m. Točka je pripeta na vzmet, ki ima neutralno dolžino $R/2$ in prožnostni koeficient k.

   (a) Določi k tako, da bo v ravnovesnem položaju materialna točka v oddaljenosti $2R/3$ od središča.

   ✓ (b) Zapiši Newtonovo enačbo v relativnem koordinatnem sistemu, ki se vrti hkrati s ploščo.

   (c) Določi frekvenco gibanja.



$$m\vec{a}_{rel} = \vec{F}_{rel} = -k\left(\rho_1 - \frac{R}{2}\right)\vec{\varepsilon}_1 + S\vec{\varepsilon}_2$$

$$-m\underbrace{\vec{a}_0}_{=0} - m\underbrace{\dot{\vec{\omega}}}_{=0}\times\vec{\rho} - m\vec{\omega}\times(\vec{\omega}\times\vec{\rho}) - 2m\vec{\omega}\times\vec{v}_{rel}$$

$$\vec{\omega} = \Omega\vec{\varepsilon}_3$$

$$\vec{\omega}\times(\vec{\omega}\times\vec{\rho}) = -\Omega^2\rho_1\vec{\varepsilon}_1$$

$$\vec{\omega}\times\vec{v}_{rel} = \Omega\dot{\rho}_1\vec{\varepsilon}_2$$

$$\Omega\vec{\varepsilon}_3\times\dot{\rho}_1\vec{\varepsilon}_1 =$$

a) $$\vec{0} = -k\left(\rho_1 - \frac{R}{2}\right)\vec{\varepsilon}_1 + S\vec{\varepsilon}_2 + m\Omega^2\rho_1\vec{\varepsilon}_1$$

$S = 0$

$$k\left(\rho_1 - \frac{R}{2}\right) = m\Omega^2\rho_1 \qquad \rho_1 = \frac{2}{3}R$$

$$\frac{2}{3}R - \frac{1}{2}R = \frac{1}{6}R$$

$$\frac{1}{6}kR = m\Omega^2\frac{2}{3}R$$

$$\boxed{k = 4m\Omega^2}$$

c) $$m\ddot{\rho}_1 = -k\left(\rho_1 - \frac{R}{2}\right) + m\Omega^2\rho_1 \qquad \frac{k}{m} = \omega^2$$

$$\ddot{\rho}_1 + (\omega^2 - \Omega^2)\rho_1 = k\frac{R}{2}$$

$$\sqrt{\omega^2 - \Omega^2}$$

$$\omega > \Omega$$

$$\sqrt{\frac{k}{m}} > \Omega \; ; \quad k > m\Omega^2$$