1 Domače naloge

Naloga 1.1. Izračunaj dvojni integral

$$\iint\limits_D \frac{xy}{(1+(x^2+y^2)^2)^2}\,dx\,dy\,,$$

kjer je

$$D = \{(x,y)\in\mathbb{R}^2\,;\, x,y\geq 0\,,\ x^2+y^2\leq 4\}\,.$$

Rešitev.

Naloga 1.2. Izračunaj volumen telesa, ki ga tvori presek dveh pravokotno, centralno se sekajočih valjev radija R.

Rešitev.

Naloga 1.3. Izračunaj vztrajnostni moment okoli osi z homogenega lika D, danega z

$$D = \{(x,y)\in\mathbb{R}^2\,;\, x^2+y^2\geq x,\ \ x^2+y^2\leq 2x\}\,.$$

Rešitev.

Naloga 1.4. Izračunaj vztrajnostni moment okoli osi y homogenega lika D, danega z

$$D = \{(x,y)\in\mathbb{R}^2\,;\, x^2+y^2\geq x,\ \ x^2+y^2\leq 2x\}\,.$$

Rešitev.

Naloga 1.5. Izračunaj volumen telesa, ki ga omejujejo ploskve

$$z=y^2 \quad z=\sqrt{y} \quad x=-1 \quad x-y-z=3\,.$$

Rešitev.

NALOGA 1.6. Izračunaj volumen telesa, ki ga omejujejo ploskve

$$z = y^2 \quad z = \sqrt{y} \quad x = -1 \quad x - y - z = 3\,.$$

REŠITEV.

NALOGA 1.7. Izračunaj maso lika

$$D = \{(x, y) \in \mathbb{R}^2\,;\, |x| \le a\,,\ |y| \le b\}\,,$$

katerega gostota je dana z

$$\rho(xy) = (a - |x|)(b - |y|)\,.$$

REŠITEV.

NALOGA 1.8. Izračunaj vzrajnostni moment okoli osi y homogenega lika

$$D = \{(x, y) \in \mathbb{R}^2\,;\, (x - p)^2 + (y - q)^2 \le 2p^2\}\,.$$

REŠITEV.

NALOGA 1.9. Izračunaj površino lika D danega z

$$D = \{(x, y) \in \mathbb{R}^2\,;\, 1 \le xy \le 8\,,\ 1 \le \frac{y^2}{x} \le 8\}\,.$$

(skiciraj lik D).

Nasvet: uporabi nove spremenljivke (u, v) dane z

$$u = xy \qquad v = \frac{y^2}{x}\,.$$

REŠITEV.

NALOGA 1.10. Izračunaj volumen telesa, ki ga omejuje

$$(x^2 + y^2 + z^2)^4 = 4z^2(x^2 + y^2)\,.$$

REŠITEV.

NALOGA 1.11. Izračunaj vztrajnostni moment okoli osi x homogenega telesa, omejenega z

$$x^2 + y^2 + z^2 \le 4 \qquad 3z^2 \ge x^2 + y^2\,.$$

REŠITEV.

NALOGA 1.12. Naj bo D lik v ravnini, tako da je $x \ge 0$ za vsako točko $(x, y) \in D$. Naj bo telo T dobljeno tako, da lik zavrtimo okoli y-osi. Pokaži, da je volumen telesa T enak

$$2\pi\bar{x}\,\text{Površina}(D)\,,$$

kjer je $\bar{x}$ abscisa masnega središča lika D.

Rešitev.

Naloga 1.13. Naj bo $D = [0,1] \times [0,1]$ in z gostoto odvisno od parametra $t \geq 0$ in sicer

$$\rho_t(x,y) = \begin{cases} 2 & ; \quad y \leq tx^2 \\ 1 & ; \quad y > tx^2 \end{cases}$$

Določi položaj masnega središča $(\bar{x}, \bar{y})$ v odvisnosti od t. Kdaj je $\bar{y}$ najmanjši?

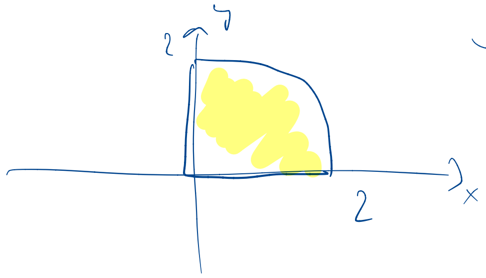
Rešitev.

REŠITVE

1.1

$$\iint_D \frac{xy}{(1+(x^2+y^2)^2)^2}\,dx\,dy \overset{\text{POLARNE KOORD.}}{=} \int_0^{\pi/2} d\varphi \int_0^2 \frac{r^2 \sin\varphi\cos\varphi}{(1+(r^2)^2)^2}\, r\,dr =$$

$D =$



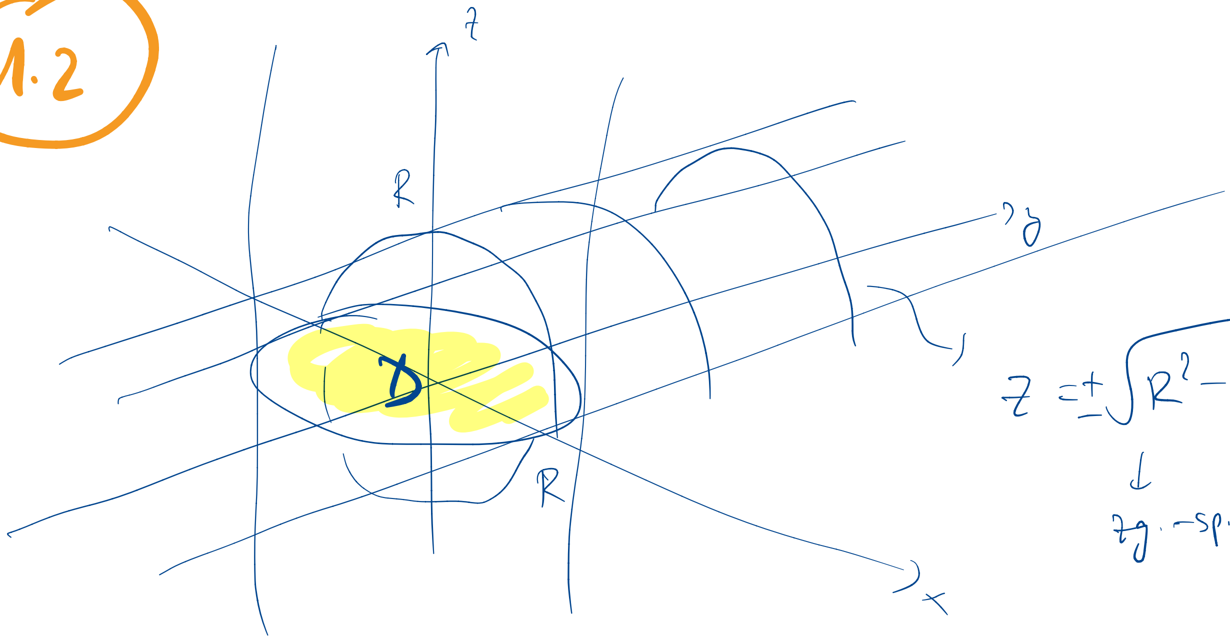
$\varphi \in [0, \pi/2]$

$r \in [0, 2]$

$$= \underbrace{\int_0^{\pi/2} \sin\varphi\cos\varphi\, d\varphi}_{= \frac{1}{2}\sin^2\varphi \Big|_0^{\pi/2} = \frac{1}{2}} \; \underbrace{\int_0^2 \frac{r^3}{(1+r^4)^2}\, dr}_{= -\frac{1}{4}(1+r^4)^{-1}\Big|_0^2} = \frac{1}{8}\left(1 - \frac{1}{33}\right)$$

$$-\frac{1}{4}(1+r^4)^{-1}\Big|_0^2 = -\frac{1}{4}(33)^{-1} + \frac{1}{4} = \frac{1}{4}\left(1 - \frac{1}{33}\right)$$

1.2



$z = \pm\sqrt{R^2 - x^2}$

$\downarrow$

zg.-sp. ploskev

$$\mathrm{Vol} = \iint_D dx\,dy \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} dz = 2\iint_D \sqrt{R^2-x^2}\,dx\,dy =$$

$$= 2\int_{-R}^{R} dx \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} \sqrt{R^2-x^2}\,dy =$$

$$= 2\int_{-R}^{R} \sqrt{R^2-x^2} \cdot 2\sqrt{R^2-x^2}\,dx = 4\int_{-R}^{R} R^2 - x^2\,dx =$$

$$= 4\left(R^2 x - x^3/3 \Big|_{-R}^{R}\right) =$$

$$= 4\left(R^3 - \frac{R^3}{3} - \left(-R^3 + R^3/3\right)\right)$$

$$= 4\left(R^3 - R^3/3 + R^3 - R^3/3\right) =$$

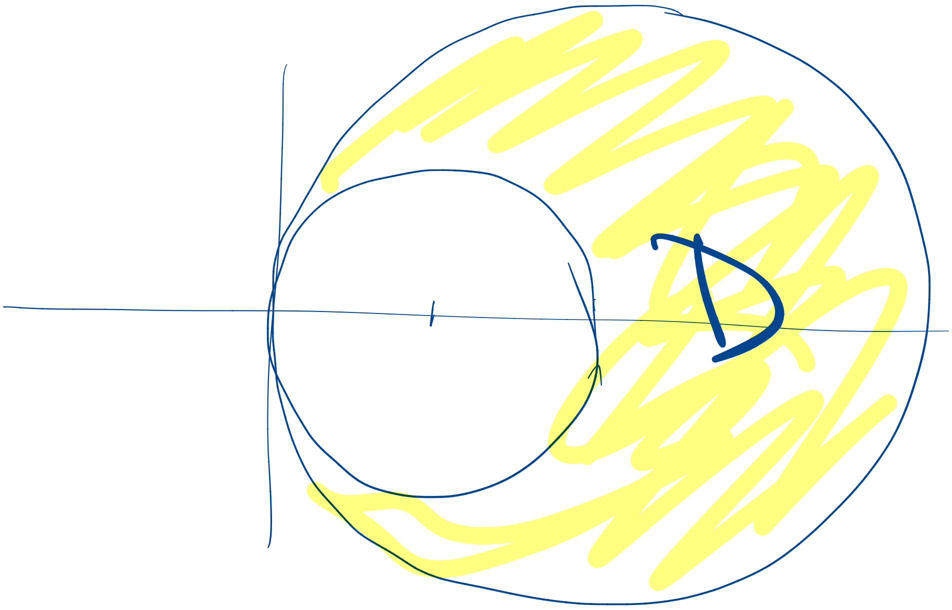
$$= 4R^3\, 4/3 = \boxed{\frac{16R^3}{3}}$$

1.3

$$x^2+y^2 \geq x \qquad\qquad x^2+y^2 \leq 2x$$

$$x^2-x+y^2 \geq 0 \qquad\qquad x^2-2x+y^2 \leq 0$$

$$(x-\tfrac{1}{2})^2+y^2 \geq \tfrac{1}{4} \qquad\qquad (x-1)^2+y^2 \leq 1$$



Uvedeme polárne koord.

$$x^2+y^2 \geq x \qquad\qquad x^2+y^2 \leq 2x$$

$$r^2 \geq r\cos\varphi \qquad\qquad r^2 \leq 2r\cos\varphi$$

$$r \geq \cos\varphi \qquad\qquad r \leq 2\cos\varphi$$

$$J_z = \iint_D x^2+y^2 \, dx\,dy = \int_{-\pi/2}^{\pi/2} d\varphi \int_{\cos\varphi}^{2\cos\varphi} r^2 \cdot r\,dr =$$

$$= 2\int_0^{\pi/2} \left(\frac{r^4}{4} \Big|_{\cos\varphi}^{2\cos\varphi} \right) d\varphi =$$

$$= 2\int_0^{\pi/2} \left(\frac{16\cos^4\varphi}{4} - \frac{\cos^4\varphi}{4} \right) d\varphi =$$

$$= \frac{15}{2}\int_0^{\pi/2} \cos^4\varphi \, d\varphi = \frac{15}{4}\, B\left(\frac{5}{2}, \frac{1}{2}\right) = \frac{15}{4}\,\frac{\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}\,\sqrt{\pi}}{\Gamma(3)} = \frac{15\cdot 3\pi}{4\cdot 2\cdot 4} = \boxed{\frac{45\pi}{32}}$$

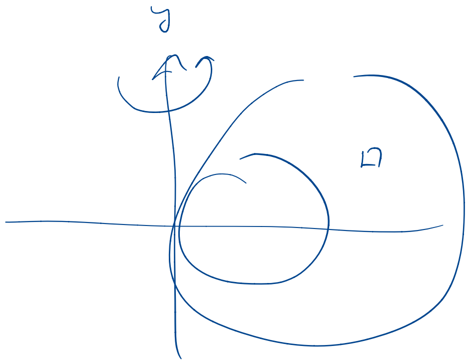
$2a-1=4$

$2b-1=0$

1.4.

Kot (1.3), le dalje

$$J_y = \iint_D x^2\,dx\,dy = \int_{-\pi/2}^{\pi/2} d\varphi \int_{\cos\varphi}^{2\cos\varphi} r^2\cos^2\varphi \, r\, dr = \ldots$$



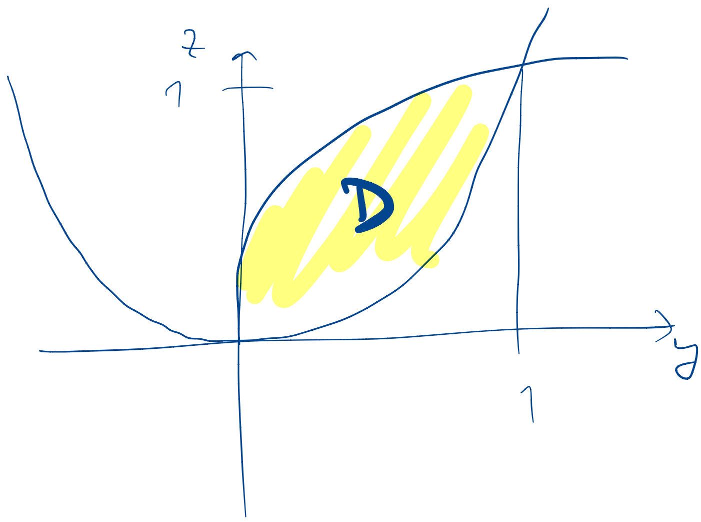
1.5

$z = y^2$

$z = \sqrt{y}$

$x = -1$

$x - y - z = 3$



$x = 3 + y + z$

$x = -1$ $\qquad -1 \le 3 + y + z \Rightarrow y + z \geq -4$ ✓

$$\mathrm{Vol} = \iint_D dy\,dz \int_{-1}^{3+y+z} dx =$$

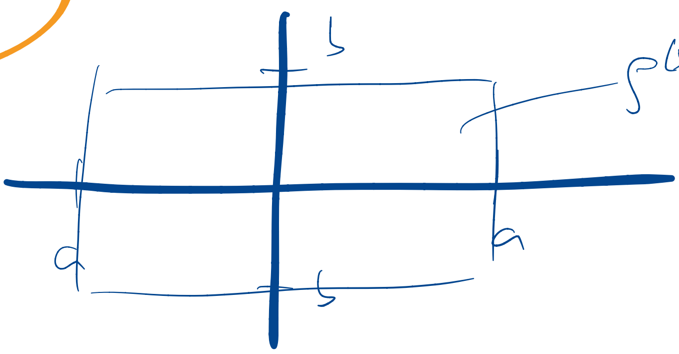
$$= \int_0^1 dy \int_{y^2}^{\sqrt{y}} dz \int_{-1}^{3+y+z} dx = \int_0^1 dy \int_{y^2}^{\sqrt{y}} 4+y+z \, dz =$$

$$= \int_0^1 dy \left(4z + yz + \frac{z^2}{2} \Big|_{y^2}^{\sqrt{y}} \right) =$$

$$= \int_0^1 \left(4\sqrt{y} + y\sqrt{y} + \frac{y}{2} - 4y^2 - y^3 - \frac{y^4}{2} \right) dy = \ldots\ldots$$

1.6 (NAPAKA ... ista kot **1.5**)

1.7



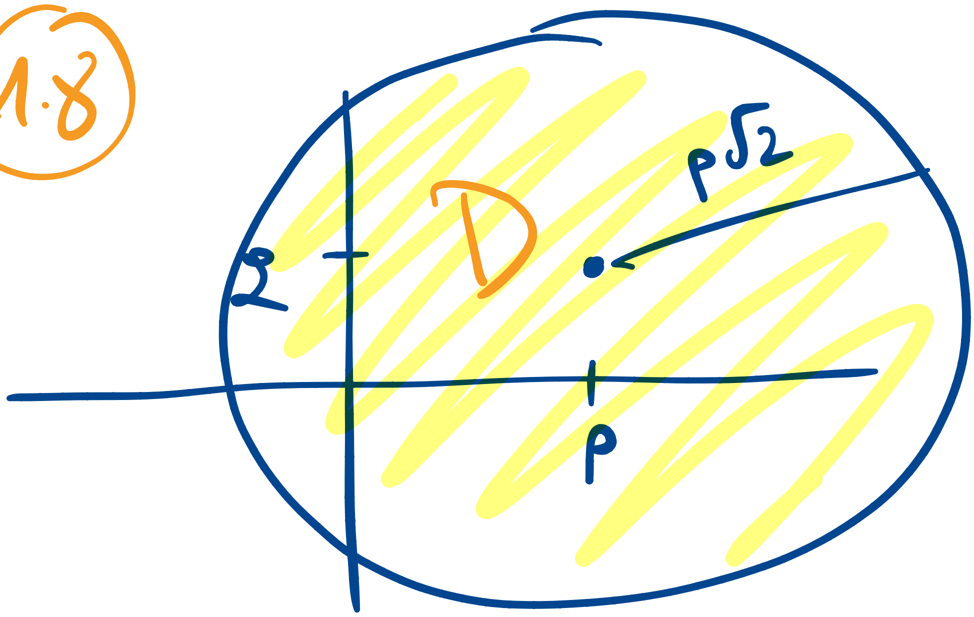
$$\rho(x,y) = (a-|x|)(b-|y|)$$

$$\text{MASA} = 4 \int_0^a dx \int_0^b (a-x)(b-y)\,dy =$$

$$= 4\int_0^a \underbrace{(a-x)\,dx}_{ax - x^2/2} \int_0^b \underbrace{(b-y)\,dy}_{by - y^2/2} = 4\left(a^2 - \frac{a^2}{2}\right)\left(b^2 - \frac{b^2}{2}\right) =$$

$$= \underline{\underline{a^2 b^2}}$$

1.8



$$J_y = \iint_D x^2\,dx\,dy = \circledast$$

$$x = p + r\cos\varphi \qquad \varphi \in [0, 2\pi]$$

$$y = \varrho + r\sin\varphi \qquad r \in (0, p\sqrt{2}] \qquad \text{Jac.} = r$$

$$\circledast = \int_0^{2\pi} d\varphi \int_0^{p\sqrt{2}} (p + r\cos\varphi)^2\, r\,dr =$$

$$= \int_0^{2\pi} d\varphi \int_0^{p\sqrt{2}} rp^2 + 2pr^2\cos\varphi + r^3\cos^2\varphi\,dr =$$

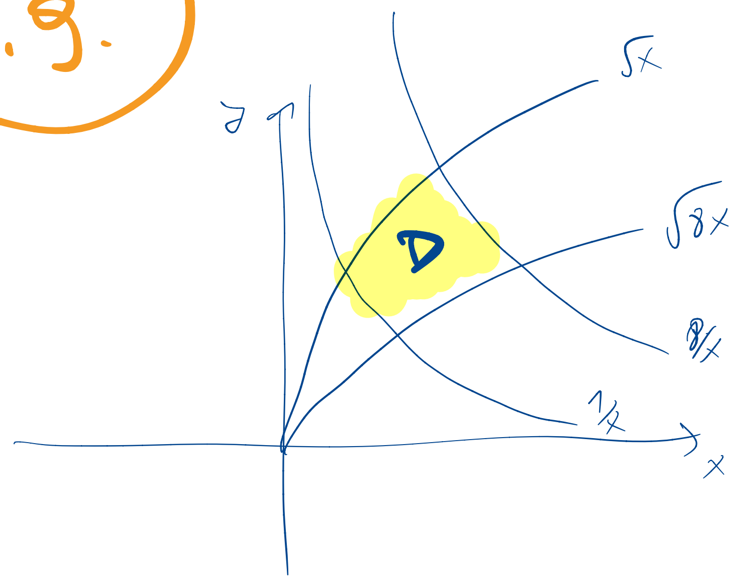
$$= \int_0^{2\pi} \left(\frac{r^2}{2}p^2 + 2p\frac{r^3}{3}\cos\varphi + \frac{r^4}{4}\cos^2\varphi \Big|_0^{p\sqrt{2}} \right) d\varphi =$$

$$= \int_0^{2\pi} \left(p^4 + \cancel{2p\frac{2\sqrt{2}p^3}{3}\cos\varphi} + p^4\cos^2\varphi \right) d\varphi =$$

$$\frac{\varphi + \cos 2\varphi}{2}\Big|_0^{2\pi} \qquad \frac{2\pi}{2} +$$ [illegible]

$$= 2\pi p^4 + p^4\pi = \boxed{3\pi p^4}$$

1.9.



$$1 \le xy \le 8$$

$$1 \le \frac{y^2}{x} \le 8$$

$$\frac{1}{x} \le y \le \frac{8}{x}$$

$$x \le y^2 \le 8x$$

$$u = xy$$
$$v = \frac{y^2}{x}$$

$$x = vy^2$$

$$u = vy^3$$

$$y = \sqrt[3]{\frac{u}{v}}$$

$$x = \frac{u}{y}$$

$$x = \sqrt[3]{\frac{u^3 v}{u}}$$

$$x = \sqrt[3]{u^2 v}$$

$$\boxed{\begin{aligned} x &= \sqrt[3]{u^2 v} = u^{2/3} v^{1/3} \\ y &= \sqrt[3]{\frac{u}{v}} = u^{1/3} v^{-1/3} \end{aligned}}$$

Je obojna v I. KVADR.

Jacobi

$$\begin{bmatrix} \frac{2}{3} u^{-1/3} v^{1/3} & \frac{1}{3} u^{2/3} v^{-2/3} \\ \frac{1}{3} u^{-2/3} v^{-1/3} & -\frac{1}{3} u^{1/3} v^{-4/3} \end{bmatrix} \xrightarrow{\det} -\frac{2}{9} v^{-1} - \frac{1}{9} v^{-1} =$$

$$= -\frac{1}{3} v^{-1} \Rightarrow \frac{1}{3v} \quad \text{ABS.}$$

$$\text{Pov.} = \iint_D dx\,dy = \int_1^8 du \int_1^8 \frac{1}{3v}\,dv =$$

(…)

$xy = u$ (gre od 1–8)

$\frac{y^2}{x} = v$ (gre od 1–8

$$= 7 \cdot \frac{1}{3} \ln v \Big|_1^8 = \frac{7}{3}(3\ln 2) = \boxed{7\ln 2}$$

1.10

Uporabimo SFERIČNE KOORD.

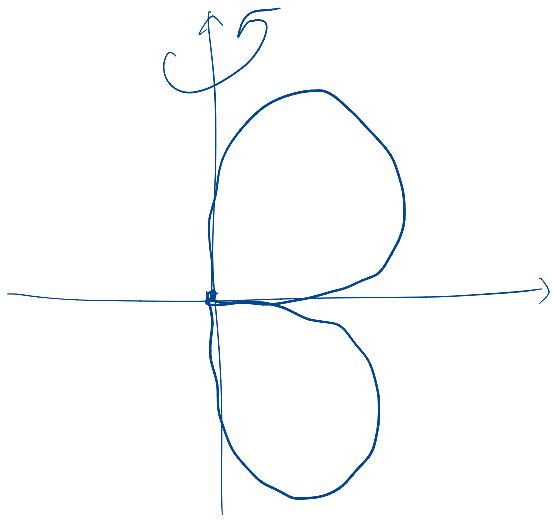
$$r^{\cancel{8}} = 4\cancel{r^2}\cos^2\vartheta\,(\cancel{r^2}\sin^2\vartheta)$$

$$r^4 = 4\sin^2\vartheta\cos^2\vartheta = (2\sin\vartheta\cos\vartheta)^2$$

$$r^4 = \sin^2 2\vartheta$$

$$r^2 = |\sin 2\vartheta|$$

$$r = \sqrt{|\sin 2\vartheta|}$$



$$\text{Vol} = \int_0^{2\pi} d\varphi \int_0^{\pi} d\vartheta \int_0^{\sqrt{|\sin 2\vartheta|}} r^2 \sin\vartheta\,dr =$$

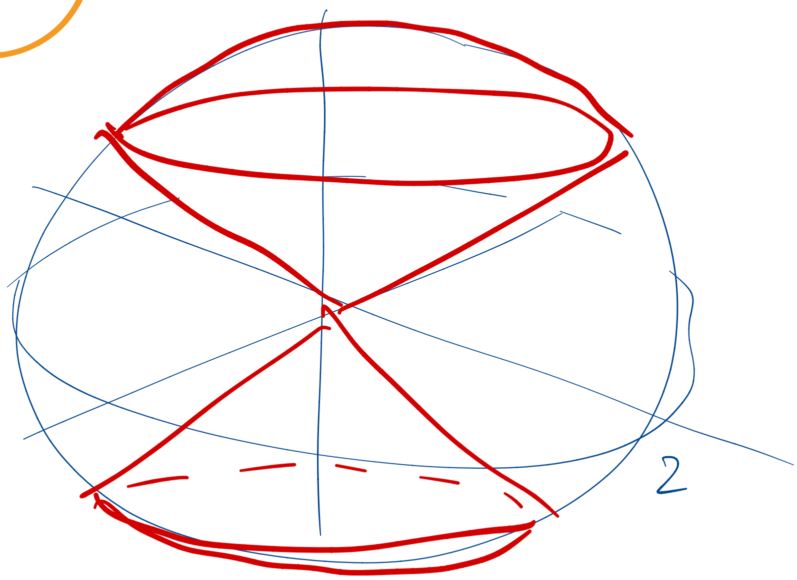
$$= 2\pi \cdot 2 \int_0^{\pi/2} \sin\vartheta \, d\vartheta \int_0^{\sqrt{\sin 2\vartheta}} r^2 \, dr =$$

$$= 4\pi \int_0^{\pi/2} \sin\vartheta \, \frac{\sin^{3/2} 2\vartheta}{3} \, d\vartheta =$$

$$= \frac{4\pi}{3} \int_0^{\pi/2} \sin\vartheta \, \sqrt{2 \sin\vartheta \cos\vartheta}^{\,3} \, d\vartheta =$$

$$= \frac{4\pi \sqrt{2}^3}{3} \int_0^{\pi/2} \cos^{3/2}\vartheta \, \sin^{5/2}\vartheta \, d\vartheta = \text{Beta FUNKCIJA} \ldots = \ldots$$

1.11



$$|z| \geq \frac{\sqrt{x^2+y^2}}{\sqrt{3}}$$

Sferične koord.

$$3z^2 \geq x^2 + y^2$$

$$3\cancel{r^2}\cos^2\vartheta \geq \cancel{r^2}\sin^2\vartheta$$

$$3 \geq \operatorname{tg}^2\vartheta$$

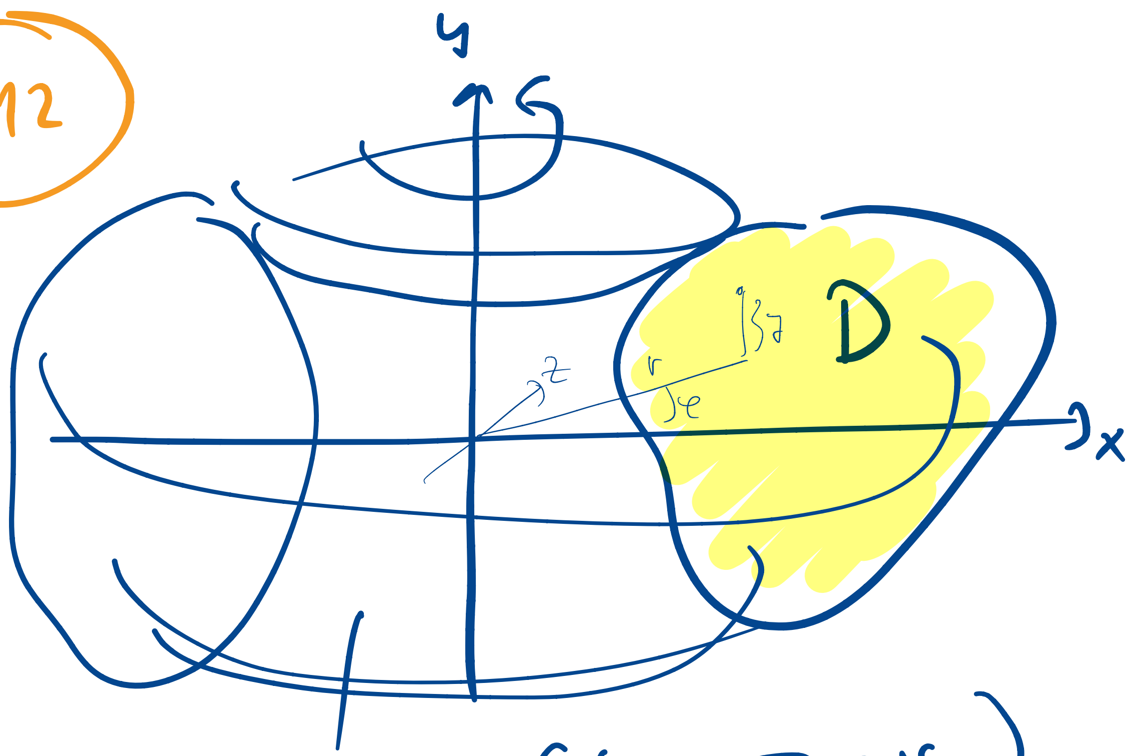
$$|\operatorname{tg}\vartheta| \leq \sqrt{3}$$

$$\vartheta \in [0, \pi/3] \cup [4\pi/6, \pi]$$

$$J_x = 2\int_0^{2\pi} d\varphi \int_0^{\pi/3} d\vartheta \int_0^2 \overbrace{r^2 \sin\vartheta}^{\text{Jac.}} \, \overbrace{r^2 \left(\sin^2\vartheta \sin^2\varphi + \cos^2\vartheta \right)}^{y^2+z^2} dr =$$

$$= 2\int_0^{2\pi} \sin^2\varphi \, d\varphi \int_0^{\pi/3} \sin^3\vartheta \, d\vartheta \int_0^2 r^4 \, dr + 2\int_0^{2\pi} d\varphi \int_0^{\pi/3} \cos^2\vartheta \sin\vartheta \, d\vartheta \int_0^2 r^4 \, dr = \dots$$

1.12



TELO (kot TORUS)

$$\mathrm{Vol} = \iiint_T dx\,dy\,dz =$$

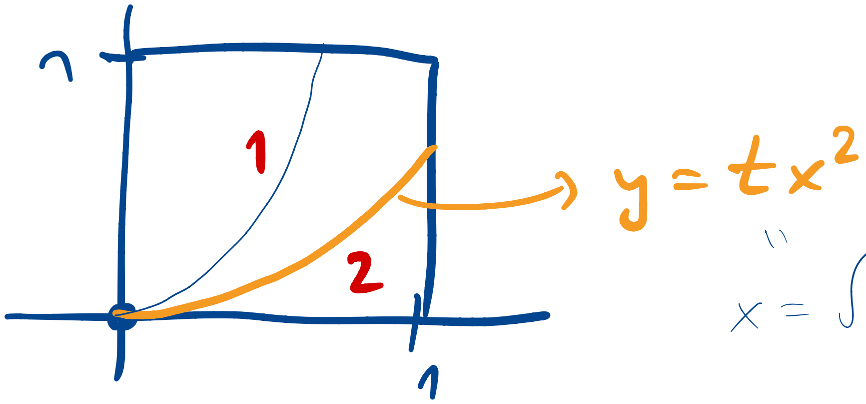
NOVE SPR. $x = r\cos\varphi$, $y = y$, $z = r\sin\varphi$ $\quad (r, y) \in D$ $\rightarrow$ Jacobi je r

$$= \int_0^{2\pi} d\varphi \iint_D r\,dr\,dz = 2\pi \iint_D r\,dr\,dz =$$

$$= 2\pi \,\mathrm{Pov}(D) \cdot \bar{r}$$

je $\bar{x}$ koordinata težišča.

1.13



$$x = \sqrt{\frac{y}{t}}$$

MASA :

$t \in [0,1]$

$$\text{MASA} = \int_0^1 dx \int_0^{tx^2} 2\,dy + \int_0^1 dx \int_{tx^2}^1 1\,dy =$$

$$= \int_0^1 2tx^2\,dx + \int_0^1 (1 - tx^2)\,dx =$$

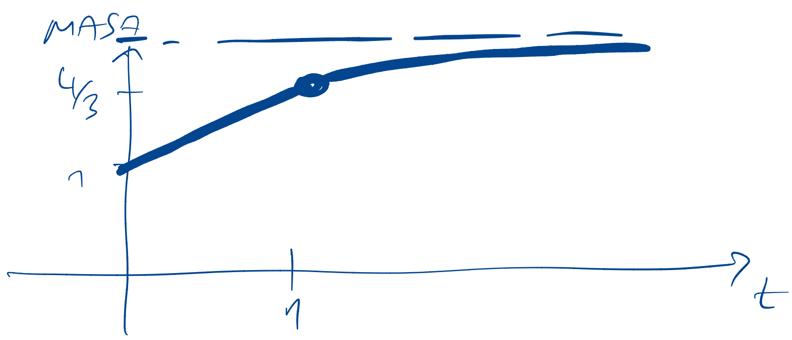
$$= \frac{2t}{3} + \left(1 - \frac{t}{3}\right) = \frac{2t}{3} + 1 - \frac{t}{3} = \boxed{1 + \frac{t}{3}}$$

$t \geq 1$

$$\text{MASA} = \int_0^1 dy \int_0^{\sqrt{y/t}} dx + \int_0^1 dy \int_{\sqrt{y/t}}^1 2\,dx =$$

$$= \int_0^1 t^{-1/2} y^{1/2}\,dy + 2\int_0^1 1 - y^{1/2} t^{-1/2}\,dy = \qquad y - t^{1/2}\frac{2}{3}y$$

$$= t^{-1/2}\frac{2}{3} + 2\left(1 - \frac{2}{3}t^{-1/2}\right) = \boxed{2 - \frac{2}{3}\frac{1}{\sqrt{t}}}$$



Če pogledamo še za $t \in [0,1]$:

$$MASA = 1 + \frac{t}{3}$$

$$\bar{y} = \frac{1}{1+t/3}\left(\int_0^1 dx \int_0^{tx^2} 2y\,dy + \int_0^1 dx \int_{tx^2}^1 y\,dy\right) =$$

$$= \frac{1}{1+t/3}\left(\frac{1}{2} + \frac{t^2}{10}\right) \Rightarrow \bar{y}(t) = \frac{5+t^2}{10(1+t/3)}$$

$$t \in [0,1]$$

$\vdots$

Podobno za $t \in [1,\infty)$

in podobno za $\bar{x}$

$\vdots$