Ponovimo Euler-Lagrangeovo enačbo

Dana je funkcija L treh spremenljivk in dve točki (a, A) in (b, B)

Iščemo tako funkcijo $y(x)$, ki gre skozi ti dve točki in ki minimizira funkcional

$$\Phi(y) = \int_a^b L(x, y(x), y'(x))\, dx$$

Pokazali smo, da je potreben pogoj za lokalni ekstrem je Euler-Lagrangeova diferencialna enačba:

$$\frac{\partial L}{\partial y}(x, y(x), y'(x)) = \frac{d}{dx}\frac{\partial L}{\partial y'}(x, y(x), y'(x))$$

Na kratko to napišemo takole

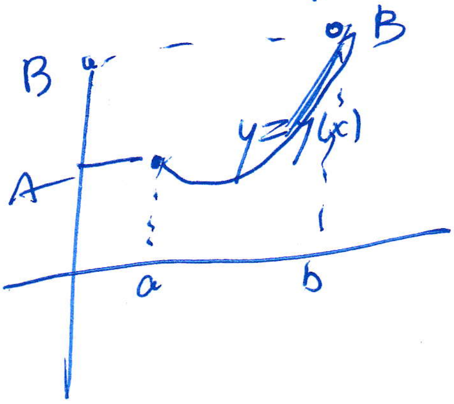
$$\frac{\partial L}{\partial y} = \frac{d}{dx}\frac{\partial L}{\partial y'}$$

Op. E.L. enačba je NDE 2. reda
V posebnih primerih jo lahko reduciramo na NDE 1. reda, če mo če $L(x, y, y')$ ni odvisna od x ali ni odvisna od y.

Primeri uporabe E.L. enačbe

Primer 1: Poišči tako funkcijo, ki gre skozi dve točki in ki ima najkrajšo dolžino

$$L = \sqrt{1+y'^2} \Rightarrow \Phi(y) = \underbrace{\int_a^b \sqrt{1+y'^2}\,dx}_{\text{dolžina krivulje}}$$



E.L. enačba se glasi

$$\frac{\partial L}{\partial y} = \frac{d}{dx}\frac{\partial L}{\partial y'}$$

$$0 = \frac{d}{dx}\frac{y'}{\sqrt{1+y'^2}}$$

Odtod sledi $\frac{y'}{\sqrt{1+y'^2}}$ konstantna, torej je tudi y' konstantna, torej je y linearna

$y(x) = cx + d$, kjer c in d določimo iz pogojev $y(a)=A$ in $y(b)=B$

Primer 2: Poišči tako funkcijo $y=y(x)$, ki gre skozi dani dve točki in katere graf da pri vrtenju okrog x osi najmanjšo površino.

Formula za površino vrtenine je

$$\int_a^b \underbrace{2\pi}_{\uparrow} \underbrace{y\sqrt{1+y'^2}}_{L}\,dx$$

ni zanimivo

Prvi poskus (neoptimalno)

$L = y\sqrt{1+y'^2}$ vstavimo v $\frac{\partial L}{\partial y} = \frac{d}{dx}\frac{\partial L}{\partial y'}$

$$\frac{\partial L}{\partial y} = \sqrt{1+y'^2}$$

$$\frac{\partial L}{\partial y'} = y \cdot \frac{1}{\cancel{2}\sqrt{1+y'^2}} \cdot \cancel{2}y' = \frac{yy'}{\sqrt{1+y'^2}}$$

$$\frac{d}{dx}\frac{\partial L}{\partial y'} = \frac{d}{dx}\left(\frac{yy'}{\sqrt{1+y'^2}}\right) = \left(y \cdot y' \cdot (1+y'^2)^{-\frac{1}{2}}\right)'$$

$$= y' \cdot y' \cdot (1+y'^2)^{-1/2} + y \cdot y'' (1+y'^2)^{-1/2}$$

$$+ y \cdot y' \cdot \left(-\frac{1}{\cancel{2}}\right)(1+y'^2)^{-3/2} \cdot \cancel{2}y'$$

$$\overset{E.L.}{=} \sqrt{1+y'^2} = (1+y'^2)^{+1/2}$$

Pomnožimo obe strani s $\sqrt{1+y'^2}$

$$\cancel{y'^2} + yy'' - \frac{yy'^2}{1+y'^2} = 1 + \cancel{y'^2}$$

$$yy''(1+y'^2) - yy'^2 = 1+y'^2$$

Tako NDE se rešuje z nastavkom

$$y' = z(y), \quad y'' = z'(y)\,y' = z'(y)\,z(y)$$

Drugi poskus (bolj optimalno)

Najprej opazimo, da $L = y\sqrt{1+y'^2}$ ni odvisna od x, zato lahko reduciramo red E.L. enačbe z naslednjim trikom

$$\frac{d}{dx}\left(L - y'\frac{\partial L}{\partial y'}\right) = \frac{d}{dx}L - \frac{d}{dx}\left(y'\frac{\partial L}{\partial y'}\right)$$

$$= \frac{\partial L}{\partial y}\cdot y' + \cancel{\frac{\partial L}{\partial y'}\cdot y''} - \cancel{y''\frac{\partial L}{\partial y'}} - y'\frac{d}{dx}\frac{\partial L}{\partial y'}$$

$$= y'\left(\frac{\partial L}{\partial y} - \frac{d}{dx}\frac{\partial L}{\partial y'}\right) \underset{E.L.}{=} y'\cdot 0 = 0$$

Odtod sledi, da je $\boxed{L - y'\frac{\partial L}{\partial y'} = C \quad (*)}$

Smisel: To je NDE 1. reda

Vstavimo v $(*)$ $L = y\sqrt{1+y'^2}$ in dobimo

$$y\sqrt{1+y'^2} - y'\cdot y\frac{y'}{\sqrt{1+y'^2}} = C \quad \Big\} \sqrt{1+y'^2}$$

$$y(1+y'^2) - yy'^2 = C\sqrt{1+y'^2}$$

$$y = C\sqrt{1+y'^2}$$

$$\frac{y^2}{C^2} = 1+y'^2$$

$$y' = \sqrt{\frac{y^2}{C^2} - 1}$$

To je NDE 1. reda z ločljivimi spr.

$$\frac{dy}{dx} = \sqrt{\frac{y^2}{C^2} - 1}$$

$$\frac{dy}{\sqrt{\frac{y^2}{C^2} - 1}} = dx$$

$$\int \frac{dy}{\sqrt{\frac{y^2}{C^2} - 1}} = \underbrace{\int dx}_{x+d}$$

subst $y = C\,\mathrm{ch}\,t$

$dy = C\,\mathrm{sh}\,t\,dt$

$\frac{y^2}{C^2} - 1 = (\mathrm{ch}\,t)^2 - 1 = (\mathrm{sh}\,t)^2$

$$\int \frac{dy}{\sqrt{\frac{y^2}{C^2} - 1}} = \int \frac{C\,\cancel{\mathrm{sh}\,t}\,dt}{\cancel{\mathrm{sh}\,t}}$$

$$= Ct$$

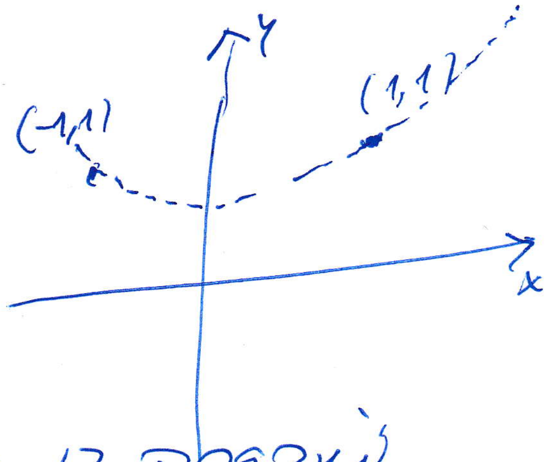
$$\overset{y=C\,\mathrm{ch}\,t}{=} C\,\mathrm{arch}\,\frac{y}{C}$$

iz $C\,\mathrm{arch}\,\frac{y}{C} = x + d$ izrazimo y

$$\mathrm{arch}\,\frac{y}{C} = \frac{x+d}{C}$$

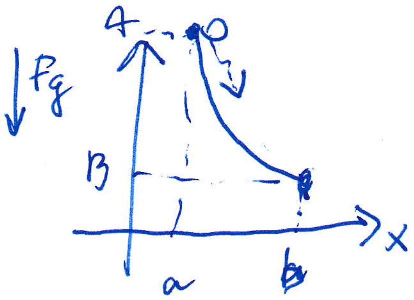
$$\frac{y}{C} = \mathrm{ch}\,\frac{x+d}{C}$$

$$y = C\,\mathrm{ch}\,\frac{x+d}{C}$$



Konstanti C in D določimo iz pogojev $y(a) = A$ in $y(b) = B$

Primer 3 Brahistohrona



$$\Phi(y) = \int_a^b \frac{\sqrt{1+y'^2}}{\sqrt{2gy}}\,dx = \min$$

$$y(a) = A, \quad y(b) = B$$

Vzamemo $L = \dfrac{\sqrt{1+y'^2}}{\sqrt{y}}$

Ker L ne vsebuje x, lahko uporabimo reducirano E. L. enačbo $L - y'\dfrac{\partial L}{\partial y'} = C$

Vstavimo $L = \dfrac{\sqrt{1+y'^2}}{\sqrt{y}}$ in dobimo

$$\frac{\sqrt{1+y'^2}}{\sqrt{y}} - y'\frac{1}{\sqrt{y}}\frac{y'}{\sqrt{1+y'^2}} = C \quad \Big\}\sqrt{y}\sqrt{1+y'^2}$$

$$1+\cancel{y'^2} - \cancel{y'^2} = C\sqrt{y}\sqrt{1+y'^2}$$

$$\sqrt{1+y'^2} = \frac{1}{C\sqrt{y}}$$

$$1+y'^2 = \frac{1}{C^2 y}$$

$$y'^2 = \frac{1}{C^2 y} - 1$$

$$y' = \sqrt{\frac{1}{C^2 y} - 1}$$

To je NDE 1. reda z ločljivimi spr.

$$\frac{dy}{dx} = \sqrt{\frac{1}{c^2 y} - 1}$$

$$\frac{dy}{\sqrt{\frac{1}{c^2 y} - 1}} = dx$$

$$\int \frac{dy}{\sqrt{\frac{1}{c^2 y} - 1}} = \int dx = x + d$$

$$\| $$

$$\int \frac{\sqrt{c^2 y}\, dy}{\sqrt{1 - c^2 y}} \qquad \text{subst} \quad y = \frac{1}{2c^2}(1-\cos\varphi)$$

$$= \frac{1}{2c^2}\, 2 \cdot \left(\sin\frac{\varphi}{2}\right)^2$$

$$= \frac{1}{c^2}\left(\sin\frac{\varphi}{2}\right)^2$$

$$\|$$

$$\int \frac{\sqrt{\left(\sin\frac{\varphi}{2}\right)^2}\, \frac{1}{c^2}\, \cancel{2} \sin\frac{\varphi}{2}\cos\frac{\varphi}{2}\, \frac{1}{\cancel{2}}\, d\varphi}{\sqrt{1 - \left(\sin\frac{\varphi}{2}\right)^2}}$$

$$\|$$

$$\frac{1}{c^2}\int \frac{\left(\sin\frac{\varphi}{2}\right)^2 \cancel{\cos\frac{\varphi}{2}}\, d\varphi}{\cancel{\cos\frac{\varphi}{2}}}$$

$$\|$$

$$\frac{1}{2c^2}\int (1-\cos\varphi)\, d\varphi$$

$$\|$$

$$\frac{1}{2c^2}(\varphi - \sin\varphi)$$

Povzetek

$$x = \frac{1}{2c^2}(\varphi - \sin\varphi) - d$$

$$y = \frac{1}{2c^2}(1-\cos\varphi)$$

Izkaže se, da je to ravno cikloida.

Konstanti C, D določim iz $y(a) = A$ in $y(b) = B$

Primer 4 : Uporaba v fiziki

Lega delca v trenutku t naj bo $y(t)$.

Kinetična energija delca je

$$T = \frac{1}{2} m v^2 = \frac{1}{2} m y'^2 .$$

Potencialna energija delca je

$$V = V(t, y)$$

Langrangeova funkcija je

$$L = T - V \quad \text{(pozor minus!!)}$$

Radi bi minimizirali funkcional

$$\Phi(y) = \int_a^b L(t, y(t), y'(t))\, dt$$

pri dodatnih pogojih

E.L. enačba se glasi $\frac{\partial L}{\partial y} = \frac{d}{dt} \frac{\partial L}{\partial y'}$

Vstavimo $L = T - V = \frac{1}{2} m y'^2 - V(t, y)$

$$\frac{\partial L}{\partial y} = -\frac{\partial V}{\partial y}$$

$$\frac{\partial L}{\partial y'} = m y' \Rightarrow \frac{d}{dt} \frac{\partial L}{\partial y'} = \frac{d}{dt}(m y') = m y''$$

Dobimo $\underbrace{-\frac{\partial V}{\partial y}}_{\text{sila na delec}} = m \underbrace{y''}_{\text{pospešek}}$ (2. Newtonov zakon)

Če potencialna energija ni odvisna od t

$V = V(y)$

potem lahko E.L. enačbo reduciram

$$L - y' \frac{\partial L}{\partial y'} = C$$

$$L = T - V = \frac{1}{2} m y'^2 - V(y)$$

$$\frac{\partial L}{\partial y'} = m y'$$

$$\frac{1}{2} m y'^2 - V(y) - m y'^2 = C$$

$$-V(y) - \frac{1}{2} m y'^2 = C$$

$$\frac{1}{2} m y'^2 + V(y) = -C$$

Vsota kinetične in potencialne energije je konstantna.

$$\underbrace{y' \frac{\partial L}{\partial y'} - L}_{\text{hamiltonova funkcija}} = \underbrace{-C}_{\text{konst}}$$

o) Izoperimetrični problem

To je vezani ekstrem za funkcionale
Imamo dva funkcionala Φ in Ψ
Iščemo minimum $\Phi(y)$ pri pogoju,
da $\Psi(y) = const$

Najprej to iz funkcionalov
prevedemo na funkcije
Vzamemo funkciji h_1 in h_2 in def.
$F(\varepsilon_1, \varepsilon_2) = \Phi(y + \varepsilon_1 h_1 + \varepsilon_2 h_2)$
$G(\varepsilon_1, \varepsilon_2) = \Psi(y + \varepsilon_1 h_1 + \varepsilon_2 h_2)$
Torej moramo rešiti vezani ekstrem
$F(\varepsilon_1, \varepsilon_2) = \min$ pri pogoju $G(\varepsilon_1, \varepsilon_2) = const.$
To se prevede na ekstreme funkcij
treh spremenljiv $\varepsilon_1, \varepsilon_2, \lambda$.
$F(\varepsilon_1, \varepsilon_2) - \lambda(G(\varepsilon_1, \varepsilon_2) - const)$

Dobiš
$$\left.\begin{aligned} \frac{\partial F}{\partial \varepsilon_1} &= \lambda \frac{\partial G}{\partial \varepsilon_1} \\ \frac{\partial F}{\partial \varepsilon_2} &= \lambda \frac{\partial G}{\partial \varepsilon_2} \\ G &= const \end{aligned}\right\}$$
Sistem treh enačb za $\varepsilon_1, \varepsilon_2, \lambda$

Omejimo se na poseben tip funkcionalov

$$\Phi(y) = \int_a^b L(x, y(x), y'(x))\,dx$$

$$\Psi(y) = \int_a^b K(x, y(x), y'(x))\,dx$$

V definicijski območji teh funkcionalov veljavno se pogoja $y(a)=A$, $y(b)=B$.

Vzemimo taki funkciji h_1 in h_2, ki zadoščata $h_1(a) = h_2(a) = 0$ in $h_1(b) = h_2(b) = 0$

Funkcija $\tilde{y} = y + \varepsilon_1 h_1 + \varepsilon_2 h_2$ torej zadošča pogojem $\tilde{y}(a) = A$ in $\tilde{y}(b) = B$.

Definirajmo

$$F(\varepsilon_1, \varepsilon_2) = \Phi(\tilde{y}) = \int_a^b L(x, y+\varepsilon_1 h_1 + \varepsilon_2 h_2, y' + \varepsilon_1 h_1' + \varepsilon_2 h_2')\,dx$$

$$G(\varepsilon_1, \varepsilon_2) = \Psi(\tilde{y}) = \int_a^b K(\quad -\!\!\shortparallel\!\!- \quad)\,dx$$

Potrebni pogoj za vezani ekstrem je potem

$$\underbrace{\frac{\partial F}{\partial \varepsilon_1} = \lambda \frac{\partial G}{\partial \varepsilon_1}}_{\text{omejimo se na tega}}, \quad \frac{\partial F}{\partial \varepsilon_2} = \lambda \frac{\partial G}{\partial \varepsilon_2}, \quad G = \text{const}$$

$$0 = \frac{\partial F}{\partial \varepsilon_1} - \lambda \frac{\partial G}{\partial \varepsilon_1} = \int_a^b \frac{\partial}{\partial \varepsilon_1} L(x, \tilde{y}, \tilde{y}')\,dx - \lambda \int_a^b \frac{\partial}{\partial \varepsilon_1} K(x, \tilde{y}, \tilde{y}')\,dx$$

$$= \int_a^b \left(\underline{\frac{\partial L}{\partial y}} h_1 + \underline{\underline{\frac{\partial L}{\partial y'}}} h_1' \right) dx - \lambda \int_a^b \left(\underline{\frac{\partial K}{\partial y}} h_1 + \underline{\underline{\frac{\partial K}{\partial y'}}} h_1' \right) dx$$

$$= ?$$

To združimo v en integral

Definiramo $M = L - \lambda K$

$$?= \int_a^b \left(\frac{\partial M}{\partial y} h_1 + \frac{\partial M}{\partial y'} h_1' \right) dx = ??$$

Upoštevamo še

$$\int_a^b \frac{\partial M}{\partial y'} h_1' \, dx = \underbrace{\frac{\partial M}{\partial y'} h_1 \Big|_a^b}_{=0 \text{ ker } h_1(a)=h_1(b)=0} - \int_a^b h_1 \frac{d}{dx} \frac{\partial M}{\partial y'} dx$$

$$= - \int_a^b h_1 \frac{d}{dx} \frac{\partial M}{\partial y'} dx$$

Torej je

$$?? = \int_a^b \left(\frac{\partial M}{\partial y} - \frac{d}{dx} \frac{\partial M}{\partial y'} \right) h_1 \, dx$$

Ta integral mora biti nič za vsak h_1 ki zadošča $h_1(a) = h_1(b) = 0$

Odtod sledi, da mora biti

$$\frac{\partial M}{\partial y} - \frac{d}{dx} \frac{\partial M}{\partial y'} = 0$$

kjer je $M = L - \lambda K$.

Koliko konstant nastopa v rešitvi enačbe

$$\frac{\partial M}{\partial y} - \frac{d}{dx}\frac{\partial M}{\partial y'} = 0$$

Ker je NDE 2. reda, imamo v rešitvi dve konstanti. Ena konstanta nastopa v definiciji $M = L - \lambda K$
↑ tretja konst.

Koliko enačb imamo za te tri konstante.

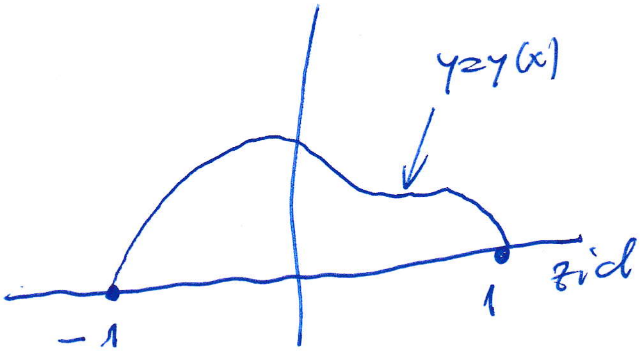
Imamo dva robna pogoja $y(a) = A$ in $y(b) = B$

In imamo še en integracijski pogoj

$$\int_a^b K(x, y, y')\,dx = \text{const.}$$

Torej imamo tri dodatne pogoje za tri neznane konstanti.

Primer: Problem kraljice Dido



Iščemo maksimum od

$$\int_{-1}^{1} y\,dx \quad (= \text{ploščina})$$

pri pogoju

$$\int_{-1}^{1} \sqrt{1+y'^2}\,dx = l \quad (= \text{fiksna dolžina})$$

Torej $L = y$, $K = \sqrt{1+y'^2}$

$$M = L - \lambda K = y - \lambda\sqrt{1+y'^2}$$

Ker v M spremenljivka x ne nastopa eksplicitno, lahko uporabimo reducirano E.L. enacbo

$$L - y'\frac{\partial L}{\partial y'} = C$$

$$L = y - \lambda\sqrt{1+y'^2}$$

$$\frac{\partial L}{\partial y'} = -\lambda\frac{y'}{\sqrt{1+y'^2}}$$

$$y - \lambda\sqrt{1+y'^2} - y'\cdot\left(-\lambda\frac{y'}{\sqrt{1+y'^2}}\right) = C$$

$$y - \lambda\sqrt{1+y'^2} + \frac{\lambda y'^2}{\sqrt{1+y'^2}} = C \;\Big\}\cdot\sqrt{1+y'^2}$$

$$y\sqrt{1+y'^2} - \lambda(1+\cancel{y'^2}) + \lambda\cancel{y'^2} = C\sqrt{1+y'^2}$$

$$(y-C)\sqrt{1+y'^2} = \lambda$$

$$1+y'^2 = \left(\frac{\lambda}{y-C}\right)^2$$

$$y' = \sqrt{\left(\frac{\lambda}{y-C}\right)^2 - 1}$$

$$\frac{dy}{dx} = \sqrt{\left(\frac{\lambda}{y-c}\right)^2 - 1}$$

$$\int \frac{dy}{\sqrt{\left(\frac{\lambda}{y-c}\right)^2 - 1}} = \int dx = x + d$$

$$\parallel$$

$$\int \frac{(y-c)\,dy}{\sqrt{\lambda^2 - (y-c)^2}}$$

$$\parallel$$

$$\int \frac{\cancel{\lambda}\cos\varphi \cdot (-\lambda \cancel{\sin\varphi})\,d\varphi}{\cancel{\lambda \sin\varphi}}$$

$$\parallel$$

$$-\lambda \int \cos\varphi \, d\varphi$$

$$\parallel$$

$$-\lambda \sin\varphi$$

substitucija

$$y = c + \lambda\cos\varphi$$

$$\lambda^2 - (y-c)^2 = \lambda^2 - (\lambda\cos\varphi)^2$$

$$= \lambda^2 (\sin\varphi)^2$$

$$dy = \lambda(-\sin\varphi)\,d\varphi$$

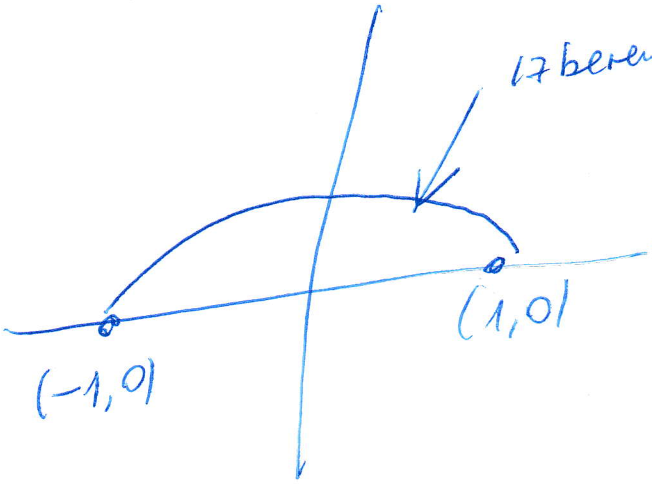
Povzetek

$$x = -d - \lambda\sin\varphi$$

$$y = c + \lambda\cos\varphi$$

To je ravno parametr. nekega kroga

konstante c, d in λ določimo iz pogojev $y(-1) = 0$, $y(1) = 0$ in $\int \sqrt{1 + y'^2}\,dx = l$



Izberemo tak krožni lok med danima točkama, ki ima dolžino l

•) Posplošitve E.L. enačbe (glej Wikipedija)

Namesto $L(x,y,y')$ bi lahko vzeli

a) $L(x, y_1, y_2, y_1', y_2')$

posplošitev na več funkcij

b) $L\left(x, y, z(x,y), \frac{\partial z}{\partial x}(x,y), \frac{\partial z}{\partial y}(x,y)\right)$

posplošitev na več neodvisnih spr.

c) $L(x,y,y',y'')$

posplošitev na višje odvode

Lahko imamo tudi kombinacije a & b & c

Tudi robne pogoje $y(a)=A$ in $y(b)=B$

lahko posplošimo ali izpustimo

(če $y(a)=A$ manjka, ga zamenjamo z $\left.\frac{\partial L}{\partial y'}\right|_a=0$)