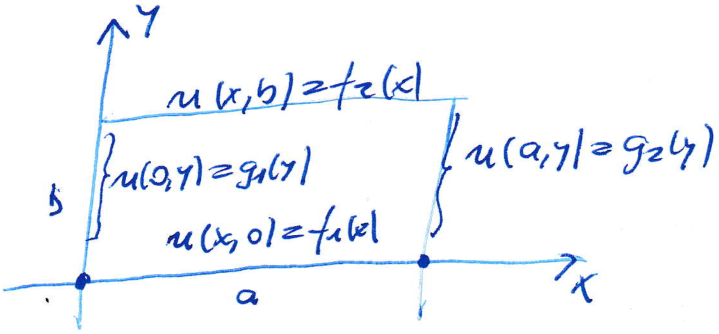
Laplaceova enačba za pravokotnik

Ponovimo od zadnjič. Imamo pravokotnik



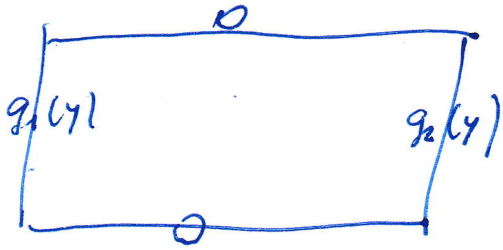
Iščemo tako funkcijo $u(x,y)$, ki znotraj pravokotnika zadošča enačbi

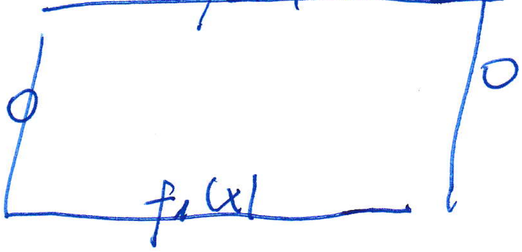
$$u_{xx}+u_{yy}=0$$

na robu pravokotnika pa ima predpisane vrednosti.

Povedali smo, da ta problem rešujemo z nastavkom $u=v+w$, kjer funkciji $v(x,y)$ in $w(x,y)$ rešita problema

$$v_{xx}+v_{yy}=0 \qquad\qquad w_{xx}+w_{yy}=0$$





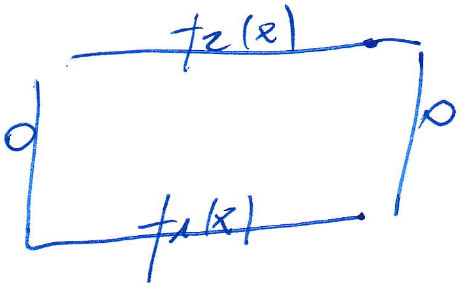
Rešili bomo drugo podnalogo.

Rešitev prve naloge je analogna.

Rešujmo nalogo ($w \to u$)

$$u_{xx}+u_{yy}=0$$

pri pogojih



$$u(0,y)=0$$
$$u(a,y)=0$$
$$u(x,0)=f_1(x)$$
$$u(x,b)=f_2(x)$$

Uporabimo Fourierovo metodo separacije spremenljivk.

Rešitev iščemo z nastavkom

$$u(x,y)=X(x)Y(y)$$

Ko ta nastavek vstavimo v $u_{xx}+u_{yy}=0$ dobimo $X''Y+XY''=0$.

Ločimo spremenljivke

$$X''Y=-XY'' \quad \} : XY$$

$$\underbrace{\frac{X''}{X}}_{\text{odvisno samo od } x} = -\underbrace{\frac{Y''}{Y}}_{\text{odvisno samo od } y} = \lambda \quad \text{konstanta}$$

Ko nastavek vstavimo v homogena robna pogoja, dobimo

$$X(0)Y(y)=0 \text{ in } X(a)Y(y)=0$$

· Če je $Y \equiv 0$, dobimo $u \equiv 0$, kar ni zanimivo.
Sicer pa dobimo $X(0) = X(a) = 0$.

Najprej rešimo enačbo

$$\frac{X''}{X} = \lambda$$

pri pogojih $X(0) = X(a) = 0$

$$X'' - \lambda X = 0$$

$$X'' + (\sqrt{-\lambda})^2 X = 0$$

$$X = A\cos(\sqrt{-\lambda}\, x) + B\sin(\sqrt{-\lambda}\, x)$$

Vstavi $x = 0$

$$0 = X(0) = A \cdot \underbrace{\cos 0}_{1} + B \underbrace{\sin 0}_{0} \Rightarrow A = 0$$

Vstavi $x = a$

$$0 = X(a) = B\sin(\sqrt{-\lambda}\, a)$$

Če $B = 0$ dobimo trivialno rešitev, kar ni zanimivo. Sicer pa dobimo

$$0 = \sin(\sqrt{-\lambda}\, a)$$

$$\Rightarrow \sqrt{-\lambda}\, a = k\pi$$

$$\Rightarrow \lambda = -\left(\frac{k\pi}{a}\right)^2 =: \lambda_k$$

Odtod sledi, da je

$$X(x) = \underset{\overset{\shortparallel}{0}}{A} \cos\left(\sqrt{-\lambda}\,x\right) + B \sin\left(\underbrace{\sqrt{-\lambda}}_{\frac{k\pi}{a}}\,x\right)$$

$$= \underbrace{B \sin \frac{k\pi x}{a}}_{X_k(x)}$$

Rešiti moramo še enačbo

$$-\frac{Y''}{Y} = \lambda = -\left(\frac{k\pi}{a}\right)^2$$

Dobimo $Y'' - \left(\frac{k\pi}{a}\right)^2 Y = 0$

Splošna rešitev je

$$Y = C \operatorname{ch}\left(\frac{k\pi y}{a}\right) + D \operatorname{sh}\left(\frac{k\pi y}{a}\right)$$

Označimo $Y_k = C_k \operatorname{ch}\left(\frac{k\pi y}{a}\right) + D_k \operatorname{sh}\left(\frac{k\pi y}{a}\right)$

Vstavimo ta ~~nastavek~~ X_k in Y_k v

$u = XY$ in dobimo

$$u(x,y) = \underbrace{\sin\left(\frac{k\pi x}{a}\right)\left(C_k \operatorname{ch}\left(\frac{k\pi y}{a}\right) + D_k \operatorname{sh}\left(\frac{k\pi y}{a}\right)\right)}_{u_k(x,y)}$$

Funkcija $u_k(x,y)$ zadošča enačbi

$$u_{xx} + u_{yy} = 0$$

in ~~robnim~~ homogenim robnim pogojem

$$u(0,y) = 0 \qquad u(a,y) = 0$$

Ne zadošča pa ta funkcija ostalima dvema robnima pogojema

$$u(x,0) = f_1(x), \quad u(x,b) = f_2(x)$$

Pomagamo si z nastavkom

$$u(x,y) = \sum_{k=1}^{\infty} u_k(x,y) \tag{*}$$

$$= \sum_{k=1}^{\infty} \sin\left(\frac{k\pi x}{a}\right)\left(C_k \operatorname{ch}\frac{k\pi y}{a} + D_k \operatorname{sh}\frac{k\pi y}{a}\right)$$

Določiti moramo še konstante C_k in D_k.

Vstavimo $y=0$ in dobimo

$$\underbrace{u(x,0)}_{f_1(x)} = \sum_{k=1}^{\infty} \sin\left(\frac{k\pi x}{a}\right)\left(\underbrace{C_k \operatorname{ch} 0 + D_k \operatorname{sh} 0}_{C_k}\right)$$

Velja $f_1(x) = \sum_{k=1}^{\infty} C_k \sin\left(\frac{k\pi x}{a}\right)$

$$\Rightarrow C_k = \underbrace{\frac{2}{a}\int_0^a f_1(x) \sin\frac{k\pi x}{a}\,dx}_{f_{1,k}}$$

- Če vstavimo $y=b$, dobimo

$$\underbrace{u(x,b)}_{f_2(x)} = \sum_{k=1}^{\infty} \sin\frac{k\pi x}{a}\left(C_k \operatorname{ch}\frac{k\pi b}{a} + D_k \operatorname{sh}\frac{k\pi b}{a}\right)$$

Odtod sledi

$$C_k \operatorname{ch}\frac{k\pi b}{a} + D_k \operatorname{sh}\frac{k\pi b}{a} = \underbrace{\frac{2}{a}\int_0^a f_2(x)\sin\frac{k\pi x}{a}\,dx}_{f_{2,k}}$$

Odtod lahko izračunamo še D_k

$$C_k = f_{1,k}$$

$$D_k = \frac{f_{2,k} - f_{1,k}\operatorname{ch}\frac{k\pi b}{a}}{\operatorname{sh}\frac{k\pi b}{a}}$$

To vstavimo v $(*)$ in dobimo

$$u(x,y) = \sum_{k=1}^{\infty} \sin\frac{k\pi x}{a}\underbrace{\left(f_{1,k}\operatorname{ch}\frac{k\pi y}{a} + \frac{f_{2,k} - f_{1,k}\operatorname{ch}\frac{k\pi b}{a}}{\operatorname{sh}\frac{k\pi b}{a}}\operatorname{sh}\frac{k\pi y}{a}\right)}_{\frac{1}{\operatorname{sh}\frac{k\pi b}{a}}\left(f_{1,k}\left(\operatorname{ch}\frac{k\pi y}{a}\operatorname{sh}\frac{k\pi b}{a} - \operatorname{ch}\frac{k\pi b}{a}\operatorname{sh}\frac{k\pi y}{a}\right) + f_{2,k}\operatorname{sh}\frac{k\pi y}{a}\right)}$$

$$\Rightarrow u(x,y) = \sum_{k=1}^{\infty} \sin\frac{k\pi x}{a}\cdot\frac{1}{\operatorname{sh}\frac{k\pi b}{a}}\left(f_{1,k}\operatorname{sh}\frac{k\pi(b-y)}{a} + f_{2,k}\operatorname{sh}\frac{k\pi y}{a}\right)$$

To je rešitev ene polovice začetne naloge.
Podobno rešimo tudi drugo polovico

•) Laplaceov operator v polarnih koordinatah

Pri reševanju Laplaceove enačbe za krog (v naslednjem razdelku) bomo prešli iz kartezičnih na polarne koordinate

$x = r\cos\varphi \qquad r = \sqrt{x^2+y^2}$

$y = r\sin\varphi \qquad \varphi = \operatorname{arctg}\frac{y}{x}$

$u(r\cos\varphi, r\sin\varphi)$
"

Uvedli bomo tudi novo funkcijo $v(r,\varphi)$, kjer je $u(x,y) = v(r,\varphi) = v(\sqrt{x^2+y^2}, \operatorname{arctg}\frac{y}{x})$

Pokazali bomo, da velja

$$\underbrace{u_{xx} + u_{yy}}_{\text{Laplaceov operator v kartezičnih koord.}} = \underbrace{v_{rr} + \frac{1}{r}v_r + \frac{1}{r^2}v_{\varphi\varphi}}_{\text{Laplaceov operator v polarnih koordinatah}}$$

$$u(x,y) = v\left(\sqrt{x^2+y^2}, \operatorname{arctg}\frac{y}{x}\right)$$

$$u_x = v_r \cdot r_x + v_\varphi \cdot \varphi_x$$

$$= v_r \frac{2x}{2\sqrt{x^2+y^2}} + v_\varphi \cdot \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \frac{-y}{x^2}$$

$$= v_r \frac{x}{\sqrt{x^2+y^2}} + v_\varphi \frac{-y}{x^2+y^2}$$

$x = r\cos\varphi$
$y = r\sin\varphi$

$$= v_r \frac{r\cos\varphi}{r} + v_\varphi \frac{-r\sin\varphi}{r^2}$$

$$= v_r\cos\varphi - v_\varphi\frac{\sin\varphi}{r} \leftarrow \text{označimo to z } w(r,\varphi)$$

Če še enkrat odvedemo po x, dobimo

$$u_{xx} = w_x = w_r \cdot \underbrace{r_x}_{\cos\varphi} + w_\varphi \cdot \underbrace{\varphi_x}_{-\frac{\sin\varphi}{r}}$$

Vstavimo $w = v_r\cos\varphi - v_\varphi \frac{\sin\varphi}{r}$

Dobimo

$$u_{xx} = \left(v_r\cos\varphi - v_\varphi\frac{\sin\varphi}{r}\right)_r \cos\varphi - \left(v_r\cos\varphi - v_\varphi \frac{\sin\varphi}{r}\right)_\varphi \frac{\sin\varphi}{r}$$

$$= \left(v_{rr}\cos\varphi - v_{\varphi r}\frac{\sin\varphi}{r} + v_\varphi\frac{\sin\varphi}{r^2}\right)\cos\varphi - \left(v_{r\varphi}\cos\varphi + v_r(-\sin\varphi) - v_{\varphi\varphi}\frac{\sin\varphi}{r} - v_\varphi\frac{\cos\varphi}{r}\right)\frac{\sin\varphi}{r}$$

Podobno dobimo

$$u_y = v_y = v_r r_y + v_\varphi \varphi_y = v_r \underbrace{\frac{2y}{2\sqrt{x^2+y^2}}}_{\frac{r\sin\varphi}{r}=\sin\varphi} + v_\varphi \underbrace{\frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\frac{1}{x}}_{\frac{x}{x^2+y^2}=\frac{r\cos\varphi}{r^2}=\frac{\cos\varphi}{r}}$$

$$= v_r\sin\varphi + v_\varphi\frac{\cos\varphi}{r}$$

Odtod sledi, da je

$$u_{yy} = u_{yr}\cdot \underbrace{r_y}_{\sin\varphi} + u_{y\varphi}\cdot\underbrace{\varphi_y}_{\frac{\cos\varphi}{r}}$$

$$= \left(v_r\sin\varphi + v_\varphi\frac{\cos\varphi}{r}\right)_r\sin\varphi + \left(v_r\sin\varphi+v_\varphi\frac{\cos\varphi}{r}\right)_\varphi\frac{\cos\varphi}{r}$$

$$= \left(v_{rr}\sin\varphi + v_{\varphi r}\frac{\cos\varphi}{r} + v_\varphi\frac{-\cos\varphi}{r^2}\right)\sin\varphi + \left(v_{r\varphi}\sin\varphi + v_r\cos\varphi + v_{\varphi\varphi}\frac{\cos\varphi}{r} + v_\varphi\frac{-\sin\varphi}{r}\right)\frac{\cos\varphi}{r}$$

Seštejemo izraza za u_{xx} in u_{yy}

$$v_{rr}\left(\left(\cos\varphi\right)^2+\left(\sin\varphi\right)^2\right)^{=1}$$

$$+\,v_r\left(\frac{(\sin\varphi)^2}{r}+\frac{(\cos\varphi)^2}{r}\right)^{=1/r}$$

$$+\,v_{\varphi r}\left(-\frac{\sin\varphi}{r}\cos\varphi-\cos\varphi\frac{\sin\varphi}{r}+\frac{\cos\varphi}{r}\cdot\sin\varphi+\sin\varphi\frac{\cos\varphi}{r}\right)^{=0}$$

$$+\,v_\varphi\left(\frac{\sin\varphi}{r^2}\cdot\cos\varphi+\frac{\cos\varphi}{r}\cdot\frac{\sin\varphi}{r}-\frac{\cos\varphi}{r^2}\cdot\sin\varphi-\frac{\sin\varphi}{r}\cdot\frac{\cos\varphi}{r}\right)^{=0}$$

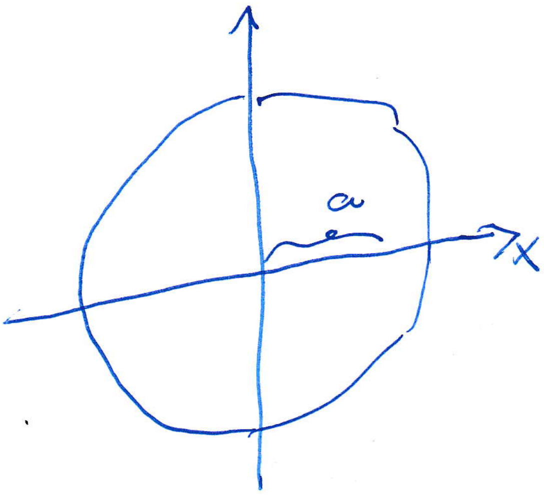
$$+\,v_{\varphi\varphi}\left(\left(\frac{\sin\varphi}{r}\right)^2+\left(\frac{\cos\varphi}{r}\right)^2\right)^{=1/r^2}$$

$$=v_{rr}\cdot 1+v_r\cdot\frac{1}{r}+v_{\varphi r}\cdot 0+v_\varphi\cdot 0+v_{\varphi\varphi}\cdot\frac{1}{r^2}$$

$$=v_{rr}+\frac{1}{r}v_r+\frac{1}{r^2}v_{\varphi\varphi}$$

To je Laplaceov operator v polarnih koordinatah.

.) Laplaceova enacba za krog



Iščemo tako funkcijo $u(x,y)$, ki znotraj kroga zadošča

$$u_{xx} + u_{yy} = 0$$

na robu pa zavzame predpisano vrednost

Najprej preidemo na polarne koordinate

$$u(x,y) = v(r,\varphi)$$

Enacba $\quad v_{rr} + \frac{1}{r} v_r + \frac{1}{r^2} v_{\varphi\varphi} = 0$

Robni pogoj $\quad v(a,\varphi) = f(\varphi) \leftarrow$ dana funkcija

Rešujemo z Fourierovo metodo separacije spremenljivk

Nastavek: $v(r,\varphi) = R(r)\Phi(\varphi)$

če to vstavimo v enacbo, dobimo

$$R''\Phi + \frac{1}{r} R'\Phi + \frac{1}{r^2} R\Phi'' = 0$$

Ločimo spremenljivke

$$R''\Phi + \frac{1}{r}R'\Phi = -\frac{1}{r^2}R\Phi'' \quad \}\cdot r^2$$

$$r^2R''\Phi + rR'\Phi = -R\Phi'' \quad \}: R\Phi$$

$$\underbrace{\frac{r^2R''}{R} + \frac{rR'}{R}}_{\text{odvisno samo od } r} = \underbrace{-\frac{\Phi''}{\Phi}}_{\text{odvisno samo od } \varphi} = \lambda$$

Dobili smo dve NDE

$$r^2R'' + rR' - \lambda R = 0$$

$$\Phi'' + \lambda\Phi = 0$$

Neznani funkciji R, Φ in neznana konstanta λ.

Rabimo še kak dodaten pogoj!

Sicer imamo premalo enačb.

Upoštevamo, da je $v(r, \varphi + 2\pi) = v(r, \varphi)$

$$R(r)\Phi(\varphi + 2\pi) = R(r)\Phi(\varphi)$$

$$\Phi(\varphi + 2\pi) = \Phi(\varphi)$$

Rešimo enačbo $\Phi'' + \lambda\Phi = 0$

pri pogoju $\Phi(\varphi + 2\pi) = \Phi(\varphi)$

splošna rešitev te enačbe je

$$\Phi(\varphi) = A\cos(\sqrt{\lambda}\,\varphi) + B\sin(\sqrt{\lambda}\,\varphi)$$

Ta funkcija ima periodo 2π natanko tedaj, ko je $\sqrt{\lambda}$ celo število

Torej velja $\sqrt{\lambda} = k \in \mathbb{N}$

$$\Rightarrow \lambda = k^2$$

Ta λ vstavimo v prvo enačbo in dobimo

$$r^2R'' + rR' - k^2R = 0$$

To je Eulerjeva LDE. Rešujemo jo z nastavkom $R(r) = r^\nu$

$$R'(r) = \nu r^{\nu-1}$$

$$R''(r) = \nu(\nu-1)r^{\nu-2}$$

Ko to vstavimo v enačbo, dobimo

$$r^2\nu(\nu-1)r^{\nu-2} + r\nu r^{\nu-1} - k^2r^\nu = 0$$

$$[\nu(\nu-1) + \nu - k^2]\,r^\nu = 0$$

$$\nu^2 - k^2 = 0 \Rightarrow \nu = \pm k$$

Splošna rešitev enačbe

$$r^2 R'' + r R' - k^2 R = 0$$

je torej $R(r) = C r^k + D r^{-k}$

Opomba: Iz fizikalnih razlogov je funkcija $v(r,\varphi)$ omejena v izhodišču. Ker je tudi Φ omejena, odtod sledi, da je funkcija $R(r)$ omejena v izhodišču.

To je možno samo v primeru, ko je $D = 0$

Povzetek: Dobili smo

$$\lambda = k^2$$

$$R(r) = C r^k$$

$$\Phi(\varphi) = A \cos k\varphi + B \sin k\varphi$$

Funkcija $v(r,\varphi) = R(r)\,\Phi(\varphi)$ zadošča enačbi in fizikalnim robnim pogojem.

Ne zadošča pa robnemu pogoju $v(a,\varphi) = f(\varphi)$

na robu kroga je $r=a$

· Preostalemu robnemu pogoju bomo poskušali zadostiti z nastavkom

$$v(r,\varphi) = \sum_{k=0}^{\infty} v_k(r,\varphi)$$

$$= \sum_{k=0}^{\infty} R_k(r)\,\Phi_k(\varphi)$$

$$= \sum_{k=0}^{\infty} r^k \left(A_k \cos k\varphi + B_k \sin k\varphi\right)$$

Na robu je $r=a$ in imamo robni pogoj $v(a,\varphi) = f(\varphi)$.

Torej je

$$f(\varphi) = v(a,\varphi) = \sum_{k=0}^{\infty} a^k \left(A_k \cos k\varphi + B_k \sin k\varphi\right)$$

To je razvoj v trigonometrijsko vrsto

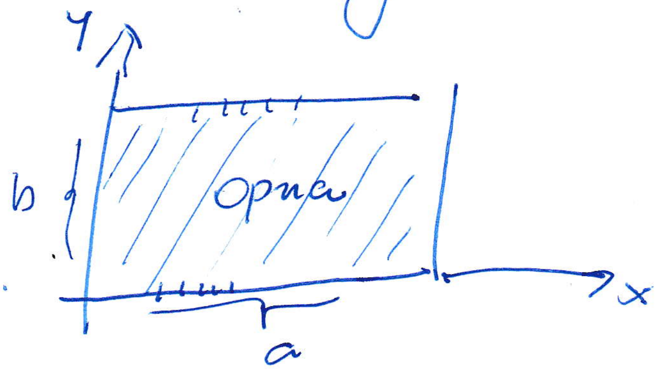
$$\left.\begin{aligned} a^k A_k &= \frac{1}{\pi}\int_0^{2\pi} f(\varphi)\cos k\varphi\, d\varphi \\ a^k B_k &= \frac{1}{\pi}\int_0^{2\pi} f(\varphi)\sin k\varphi\, d\varphi \end{aligned}\right\} k \geq 1$$

$$A_0 = \frac{1}{2\pi}\int_0^{2\pi} f(\varphi)\, d\varphi$$

To so formule za neznane koeficiente A_k in B_k

o) Nihanje pravokotne opne

To je posplošitev nihanja strune na dve dimenziji



opna je na robu pravokotnika upeta in niha transverzalno

Odmik opne od mirovne lege je
$u(x,y,t)$

Ta funkcija zadošča PDE
$$c^2(u_{xx}+u_{yy})=u_{tt}$$

robnim pogojem $u(x,0,t)=0 \quad u(0,y,t)=0$
$$u|_{\partial D}=0 \iff u(x,b,t)=0 \quad u(a,y,t)=0$$

in začetnim pogojem
$u(x,y,0)=f(x,y)$ (začetni odmik opne)
$u_t(x,y,0)=g(x,y)$ (začetna hitrost opne).

Uporabimo metodo separacije spremenljivk.
Pomagamo si z nastavkom
$$u(x,y,t)=X(x)\,Y(y)\,T(t)$$

Ko ta nastavek vstavimo v enačbo

$$c^2(u_{xx}+u_{yy})=u_{tt}$$

dobimo

$$c^2(X''YT+XY''T)=XYT'' \quad \Big\}:c^2XYT$$

$$\underbrace{\frac{X''}{X}+\frac{Y''}{Y}}_{\text{odvisno samo od } x \text{ in } y}=\underbrace{\frac{T''}{c^2T}}_{\text{odvisno samo od } t}=\lambda$$

$$\underbrace{\frac{X''}{X}}_{\text{odvisno samo od } x}=\underbrace{\frac{T''}{c^2T}-\frac{Y''}{Y}}_{\text{odvisno od } t \text{ in } y}=\mu$$

$$\underbrace{\frac{Y''}{Y}}_{y}=\underbrace{\frac{T''}{c^2T}-\frac{X''}{X}}_{t,x}=\nu$$

Velja $\lambda=\mu+\nu$

$$\frac{X''}{X}=\mu,\quad \frac{Y''}{Y}=\nu,\quad \frac{T''}{c^2T}=\lambda=\mu+\nu$$

Nastavek $u(x,y,t) = X(x)\,Y(y)\,T(t)$

vstavimo še v robne pogoje in dobimo

$X(x)Y(0)T(t) = 0$ $\qquad$ $X(0)Y(y)T(t) = 0$

$X(x)Y(b)T(t) = 0$ $\qquad$ $X(a)Y(y)T(t) = 0$

Ker nobena od X, Y, T ni identično enaka nič, lahko krajšamo in dobimo

$$Y(0) = Y(b) = 0$$

$$X(0) = X(a) = 0$$

Najprej rešimo enačbo

$$\frac{X''}{X} = \mu$$

pri pogoju $X(0) = X(a) = 0$

kot običajno dobimo, da je

$$X = A\cos(\sqrt{-\mu}\,x) + B\sin(\sqrt{-\mu}\,x)$$

vstavimo $x=0$ in $x=a$ in dobimo

$$\underbrace{X = B\sin\frac{k\pi x}{a}}_{X_k(x)} \qquad \underbrace{\mu = -\left(\frac{k\pi}{a}\right)^2}_{\mu_k}$$

Podobno rešimo drugo enačbo

$$\frac{Y''}{Y} = \nu$$

Pri pogoji $Y(0) = Y(b) = 0$

$$Y = C\cos(\sqrt{-\nu}\, y) + D\sin(\sqrt{-\nu}\, y)$$

Vstavimo $y = 0$ in $y = b$ in dobimo

$$\underbrace{Y = D\sin\frac{l\pi y}{b}}_{Y_l(y)} \qquad \underbrace{\nu = -\left(\frac{l\pi}{b}\right)^2}_{\nu_l}$$

Ostane še tretja enačba

$$\frac{T''}{c^2 T} = \lambda = \mu + \nu$$

$$T = E\cos(\sqrt{-\lambda}\, ct) + F\sin(\sqrt{-\lambda}\, ct)$$

označimo $\lambda_{k,l} = \mu_k + \nu_l = -\left(\frac{k\pi}{a}\right)^2 - \left(\frac{l\pi}{b}\right)^2$

$$T_{k,l} = E_{k,l}\cos(\sqrt{-\lambda_{k,l}}\, ct) + F_{k,l}\sin(\sqrt{-\lambda_{k,l}}\, ct)$$

Funkcija

$$u_{k,l}(x,y,t) = X_k(x)\,Y_l(y)\,T_{k,l}(t)$$
$$= \sin\frac{k\pi x}{a}\,\sin\frac{l\pi y}{b}\left(E_{k,l}\cos\left(\sqrt{\frac{k^2}{a^2}+\frac{l^2}{b^2}}\,\pi c t\right) + F_{k,l}\sin\left(\sqrt{\frac{k^2}{a^2}+\frac{l^2}{b^2}}\,\pi c t\right)\right)$$

Zadošča enačbi in robnim pogojem.

Ne zadošča pa začetnim pogojem.

Tvorimo nastavek

$$u(x,y,t) = \sum_{k=1}^{\infty}\sum_{l=1}^{\infty} u_{k,l}(x,y,t)$$

in to vstavimo v začetna pogoja

$$u(x,y,0) = f(x,y), \quad u_t(x,y,0) = g(x,y)$$

$$f(x,y) = u(x,y,0) = \sum_{k=1}^{\infty}\sum_{l=1}^{\infty} X_k(x)\,Y_l(y)\,\underbrace{T_{k,l}(0)}_{=E_{k,l}}$$

$$g(x,y) = u_t(x,y,0) = \sum_{k=1}^{\infty}\sum_{l=1}^{\infty} X_k(x)\,Y_l(y)\,\underbrace{T'_{k,l}(0)}_{=\pi c F_{k,l}\sqrt{\frac{k^2}{a^2}+\frac{l^2}{b^2}}}$$

Neznane konstante $E_{k,l}$ in $F_{k,l}$ izračunamo s pomočjo dvojne sinusne vrste

$$f(x,y) = \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} E_{k,l} \sin \frac{k\pi x}{a} \sin \frac{l\pi y}{b}$$

$$E_{k,l} = \frac{2}{a} \cdot \frac{2}{b} \underbrace{\int_0^a \int_0^b f(x,y) \sin \frac{k\pi x}{a} \sin \frac{l\pi y}{b} \, dx \, dy}_{\text{dropui integral}}$$