# LINEARNA ALGEBRA 2020/21
# 17. VAJE: 24. 2. 2021

1. Pokaži, da je preslikava $A\colon \mathbb{R}[x]_n \to \mathbb{R}[x]_n$ podana s predpisom

$$A(p(x)) = x^n p\left(\frac{1}{x}\right)$$

   linearna.

   Ali je izomorfizem? Za $n = 2$ zapiši kakšno pripadajočo matriko v standardni bazi.

2. Zapiši matriko, ki v standardni bazi pripada zrcaljenju čez premico $y = kx$.

3. Naj bo $A\colon \mathbb{R}^3 \to \mathbb{R}^3$ preslikava, ki jo dobimo z zaporedjem preslikav: razteg v smeri $(1, 2, 0)$ za 2-krat, zrcaljenje čez premico razpeto z $(0, 1, 1)$ in skrčitvijo celotnega prostora za 2-krat. Preslikavi zapiši pripadajočo matriko v standardni bazi.

4. Naj bo $L\colon \mathbb{R}^3 \to \mathbb{R}^3$, ki prostor raztegne za 3-krat v smeri $(1, 1, 1)$ in obrne v pozitivni smeri za $\pi/2$ (kaj pa za poljuben kot $\phi$) okoli osi $(1, 1, 1)$. Preslikavi zapiši pripadajočo matriko v standardni bazi.

5. Pokaži, $r(AB) \leq \min\{r(A), r(B)\}$.

6. Določi rang (in ničelnost) linearnima preslikavama $\vec{x} \mapsto \vec{a} \times \vec{x}$ in $\vec{x} \mapsto (\vec{a} \times \vec{x}) \times \vec{b}$ v odvisnosti od $\vec{a}, \vec{b} \in \mathbb{R}^3$.

7. Pokaži, da ima enačba $\langle \vec{x}, \vec{a} \rangle \vec{a} + \vec{a} \times \vec{x} = \vec{b}$ za $\vec{a} \neq 0$ natanko eno rešitev.

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Ali je izomorfizem? Za $n = 2$ zapiši kakšno pripadajočo matriko v standardni bazi.

$$A(x^2) = x^n \frac{1}{x^2} = x^{n-2} \in \mathbb{R}[x]_n$$

$$A(x^2+1) = x^n\left(\frac{1}{x^2} + 1\right) = x^{n-2} + x^n$$

$$p(x) = \sum_{i=0}^{n} a_i x^i \qquad p\left(\frac{1}{x}\right) = \sum_{i=0}^{n} a_i x^{-i}$$

$$x^n p\left(\frac{1}{x}\right) = \sum_{i=0}^{n} a_i x^{n-i} \qquad n \geq n-i \geq 0$$

A slika v $\mathbb{R}[x]_n$

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$$A(p+q) = A(p) + A(q)$$

$$x^n\left(p\left(\frac{1}{x}\right) + q\left(\frac{1}{x}\right)\right) = x^n p\left(\frac{1}{x}\right) + x^n q\left(\frac{1}{x}\right)$$

$$A(\lambda p) = x^n \lambda p\left(\frac{1}{x}\right) = \lambda x^n p\left(\frac{1}{x}\right) = \lambda A(p)$$

A je linearna

Matrika preslikave v standardni bazi

Baza $\{x^n, x^{n-1}, x^{n-2}, \dots, x, 1\}$

Vrstni red je pomemben

npr. v bazi $\{x^2, 1, x\}$ dobimo matriko $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$A(1) = x^2 \cdot 1 = x^2 = 0 \cdot 1 + 0 \cdot x + 1 x^2$

$A(x) = x^2 \cdot \frac{1}{x} = x$

$A(x^2) = x^2 \cdot \frac{1}{x^2} = 1$

$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

splošen n.

$A(x^k) = x^n \cdot \frac{1}{x^k} = x^{n-k}$

$\bar{A} e_k = e^{n-k}$

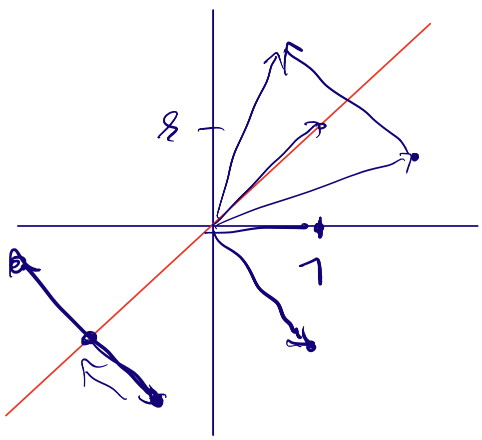
$\bar{A} \begin{bmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{bmatrix}$

A je izomorfizem (bijektivna linearna preslikava)

$\det \bar{A} = \pm 1 \neq 0 \Rightarrow \bar{A}$ je obrnljiva $\to$ A je obrnljiva

$\boxed{\begin{array}{l} A: U \to V \text{ je izomorfizem } \Leftrightarrow [A]_{B \leftarrow C} \text{ obrnljiva} \\ B, C - \text{bazi} \end{array}}$

2. Zapiši matriko, ki v standardni bazi pripada zrcaljenju čez premico $y = kx$.




$\mathbb{R}^2$

vektor na premici

$$\mathcal{A}\begin{pmatrix}1\\k\end{pmatrix} = \begin{pmatrix}1\\k\end{pmatrix}$$

$$\mathcal{A}\begin{pmatrix}0\\0\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

vektor pravokoten na premico

$$\mathcal{A}\begin{pmatrix}k\\-1\end{pmatrix} = \begin{pmatrix}-k\\1\end{pmatrix} = -\begin{pmatrix}k\\-1\end{pmatrix}$$

$$A_B = \begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}$$

$$A_S = P_{S\leftarrow B} A_B P_{B\leftarrow S} \qquad P_{S\leftarrow B} = \begin{bmatrix}1 & -k\\k & 1\end{bmatrix}$$

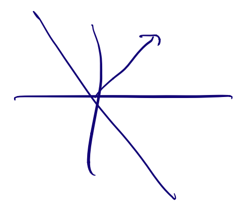
$$P_{B\leftarrow S} = P_{S\leftarrow B}^{-1} = \begin{bmatrix}1 & -k\\k & 1\end{bmatrix}^{-1} = \frac{1}{k^2+1}\begin{bmatrix}1 & k\\-k & 1\end{bmatrix}$$

$$A_S = \frac{1}{k^2+1}\begin{bmatrix}-k^2+1 & 2k\\2k & k^2-1\end{bmatrix}$$

3. Naj bo $A\colon \mathbb{R}^3 \to \mathbb{R}^3$ preslikava, ki jo dobimo z zaporedjem preslikav: razteg v smeri $(1,2,0)$ za 2-krat, zrcaljenje čez premico razpeto z $(0,1,1)$ in skrčitvijo celotnega prostora za 2-krat. Preslikavi zapiši pripadajočo matriko v standardni bazi.

- razteg v smeri $(1,2,0)$ za $2$ $\quad A_1$

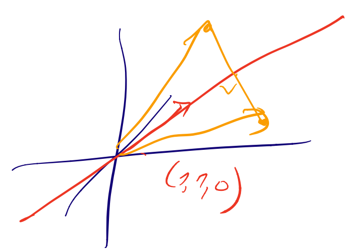
$$A_1(1,2,0) = 2(1,2,0)$$
$$A_1(2,-1,0) = (2,-1,0)$$
$$A_1(0,0,1) = (0,0,1)$$



$$\overline{A_1} = \begin{bmatrix} 2 & & \\ & 1 & \\ & & 1 \end{bmatrix}$$

- zrcaljenje čez premico $(1,1,0)$ $\quad A_2$

$$A_2(1,1,0) = (1,1,0)$$
$$A_2(-1,1,0) = -(-1,1,0)$$
$$A_2(0,0,1) = -(0,0,1)$$




$$\overline{A_2} = \begin{bmatrix} 1 & & \\ & -1 & \\ & & -1 \end{bmatrix}$$

- Celoten prostor skrčimo 2 krat

$$A_3 v = \frac{1}{2} v \qquad = \begin{bmatrix} \frac{1}{2} & & \\ & \frac{1}{2} & \\ & & \frac{1}{2} \end{bmatrix}$$

$A = A_3 A_2 A_1$ $\qquad A_S$

$\mathcal{B}_1 = \{(1,2,0), (2,-1,0), (0,0,1)\}$

$\mathcal{B}_2 = \{(1,1,0), (-1,1,0), (0,0,1)\}$

$B_S$ = katerakoli baza

$$A_S = P_{S\leftarrow B_2} \overline{A}_3 \overline{A}_2 P_{B_2 \leftarrow B_1} \overline{A}_1 P_{B_1 \leftarrow S}$$

$$P_{B_1 \leftarrow S} = \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}^{-1} & \\ & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{5} & \frac{2}{5} & \\ \frac{2}{5} & -\frac{1}{5} & \\ & & 1 \end{bmatrix}$$

$$P_{B_2 \leftarrow B_1} = P_{B_2 \leftarrow S} P_{S \leftarrow B_1} = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} & \frac{1}{2} & \\ \frac{1}{2} & -\frac{3}{2} & \\ & & 1 \end{bmatrix}$$

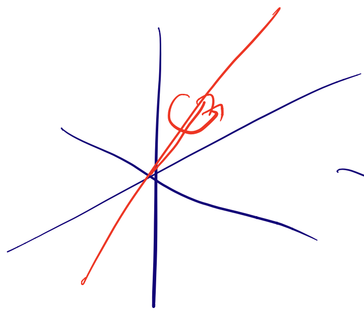
$$P_{S \leftarrow B_2} = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

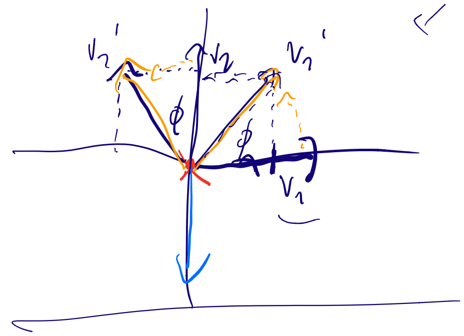
$$\begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & & \\ & \frac{1}{2} & \\ & & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & & \\ & -1 & \\ & & -1 \end{bmatrix} \begin{bmatrix} \frac{3}{2} & \frac{1}{2} & \\ \frac{1}{2} & -\frac{3}{2} & \\ & & 1 \end{bmatrix} \begin{bmatrix} 2 & & \\ & 1 & \\ & & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{5} & \frac{2}{5} & \\ \frac{2}{5} & -\frac{1}{5} & \\ & & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{5} & \frac{9}{10} & 0 \\ \frac{3}{5} & \frac{1}{5} & 0 \\ 0 & 0 & -\frac{1}{2} \end{bmatrix} = A_S$$

4. Naj bo $L\colon \mathbb{R}^3 \to \mathbb{R}^3$, ki prostor raztegne za 3-krat v smeri $(1,1,1)$ in obrne v pozitivni smeri za $\pi/2$ (kaj pa za poljuben kot $\phi$) okoli osi $(1,1,1)$. Preslikavi zapiši pripadajočo matriko v standardni bazi.

Ravnina pravokotna na $(1,1,1)$






$v_1 \perp v_2 \qquad \|v_1\| = \|v_2\|$

$\phi \to \frac{\pi}{2} - \phi$

$$v_1' = \cos\phi \cdot v_1 + \sin\phi \cdot v_2$$

$$v_2' = -\sin\phi\, v_1 + \cos\phi\, v_2 \quad \to \quad \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$$

$$\begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- $(-1,2,-1)$ $v_1 \to \dfrac{v_1}{\|v_1\|}$
- $\ominus\, v_1 \times v_2 \to \dfrac{v_1 \times v_2}{\|v_1 \times v_2\|}$
- $(1,1,1)$. $v_2$ $\quad v_2$

$B$

$$P_{S\leftarrow B} \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & \\ & & 3 \end{bmatrix} P_{B\leftarrow S}$$

$$\begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}^{-1} = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}^{T}$$

matrika je ortogonalna

5. Pokaži, $r(AB) \leq \min\{r(A), r(B)\}$.

($A, B$ matriki / preslikavi OZ)

$A: U \to V$ lin. preslikava

slika, jedro (kernel)

$\operatorname{im}(A) = \{Au \mid u \in U\} \subseteq V$, $\ker(A) = \{u \mid Au = 0\} \subseteq U$

ničelnost

$r(A) = \operatorname{rang}(A) = \dim \operatorname{im}(A)$ $\quad$ $n(A) = \dim \ker(A)$

Če je $A$ matrika, potem je $r(A)$ = št. lin neodvisnih stolpcev/vrstic

- $r(A) + n(A) = \dim(U)$
- če je $A: U \to V$ $B, C$ - bazi $r[A_{C \leftarrow B}] = r(A)$
- če je $A$ matrika: $r(A) = r(A^T)$, $P, Q$ obrnljivi matriki $r(PAQ) = r(A)$
- $n(AB) \leq n(A) + n(B)$

...

Pokažimo $r(AB) \leq \min\{r(A), r(B)\}$

(oz. $r(AB) \leq r(A)$ in $r(AB) \leq r(B)$

a) $r(AB) \leq r(A)$

$B: U \to V$

$A: V \to W$

$AB: U \to W$

$\operatorname{Im}(AB) = \{ABu \mid u \in U\} = \{Ax \mid x = Bu,\ u \in U\}$
$= \{Ax \mid x \in \operatorname{Im} B\} \subseteq \{Ax \mid x \in V\}$

Če je $U \leq V$, potem je $\dim U \leq \dim V$

b1) $r(AB) = r((AB)^T) = r(B^T A^T) \overset{\text{točka a)}}{\leq} r(B^T) = r(B)$

b2) Preslikavo $A|_{\operatorname{im} B}: \operatorname{im} B \to W$

$n(A|_{\operatorname{im}B}) + r(A|_{\operatorname{im}B}) = \dim(\operatorname{im} B) = r(B)$

$\operatorname{im}(A|_{\operatorname{im}B}) = \operatorname{im}(AB)$

$\{Ax \mid x \in \operatorname{im} B\} = \{Ax \mid x = By\ y \in U\} = \{ABy \mid y \in U\}$

$r(AB) \leq r(B)$