Deformaction retracts

Definicija 3.1 *Let $A \subset X$. A map $r\colon X \to A$ such that $r|_A = id_A$ is called a* ***retraction*** *and a subspace A is called a* ***retract*** *of X.*
A subspace $A \subset X$ is ***deformation retract****, if there exists a homotopy $H\colon X \times \mathbb{I} \to X$ from id_X and a retraction $r\colon X \to A$. A homotopy H is called* ***deformation retraction****. If the homotopy H fixes the subset A then H is called* ***strong deformation retraction*** *and a subset A is called* ***strong deformation retract*** *of the space X.*

Naloga 3.2

1. *Find all retracts of $\mathbb{S}^1$.*
2. *Find all deformation retracts of $\mathbb{S}^1$.*
3. *Find all strong deformation retracts of $\mathbb{S}^1$.*

Naloga 3.3 *Prove that deformation retractions $F, G\colon X \times \mathbb{I} \to X$ are homotopic.*

Naloga 3.4 *Let $x_0 \in X$ be strong deformation retract of X.*

1. *Prove that for every neighborhood U of the point x_0 there exists a neighborhood $V \subset U$ of x_0 such that the inclusion $i\colon V \hookrightarrow U$ is homotopy to a constant map.*
2. *Prove that X is locally connected at the point x_0.*

Naloga 3.5 *Let $X = \big(\mathbb{I} \times \{0\}\big) \cup \big(\{0\} \times \mathbb{I}\big) \cup \big(\cup_{n=1}^{\infty} \{\frac{1}{n}\} \times \mathbb{I}\big)$.*

1. *Prove that every point in X is a deformation retract of X.*
2. *Find all points in X which are strong deformation retracts of X.*

Naloga 3.6 *Let $X = ([0,1] \times \{0\}) \cup \{(x + a, \frac{x}{n}) \mid x \in [0, \frac{1}{2}], a \in \{0, \frac{1}{2}\}, n \in \mathbb{N}\}$.*

1. *Prove that X is contractible. (Hence every point in X is a deformation retract of X.)*
2. *Find all points in X which are strong deformation retracts of X.*

Naloga 3.7 *Let*

$$X = ([0,1] \times \{0\}) \cup \left\{ \left(x + a, \frac{x}{n}\right) \mid m \in \mathbb{N}, x \in \left[0, \frac{1}{2^m}\right], a = \frac{2^{m-1} - 1}{2^{m-1}}, n \in \mathbb{N} \right\}.$$

Prove that X is contractible.

Naloga 3.8 *Let*

$$T = \Big(\mathbb{I} \times \{0\}\big) \cup \big(\cup_{q \in \mathbb{Q} \cap \mathbb{I}} \{q\} \times [0, q]\big)$$

and let $\hat{T}$ be a set which is the mirror image of T over the line $x = y$ and translated by the vector $(0, -1)$. Let $X_0 = T \cup \hat{T}$ and for all $n \in \mathbb{Z}$ let X_n be the set X_0 translated by the vector (n, n). Let $X = \cup_{n \in \mathbb{Z}} X_n$.

1. *Prove that every point in X is a deformation retract of X.*
2. *Prove that there are no point in X which is a strong deformation retract of X.*