Homotopy

Naloga 1.1 *Let X be a topological space.*

1. *Prove taht $f, g\colon X \to \mathbb{R}^n$ are homotopic.*
2. *Let $r > 0$. Prove that $f, g\colon X \to \{(x, y) \in \mathbb{R}^2 \mid |x|^r + |y|^r \leq 1\}$ are homotopic.*
3. *A subset $A \subset \mathbb{R}^n$ is **star-shaped** if there exists $a \in A$ such that for every $x \in A$ the segment $\{(1-t)a + tx \mid t \in \mathbb{I}\}$ is in the set A. Prove that $f, g\colon X \to A$ are homotopic, provided A is star-shaped subset of $\mathbb{R}^n$.*
4. *Prove that $f, g\colon X \to \mathbb{S}^2 \setminus \{(0,0,1)\}$ are homotopic.*

Naloga 1.2 *Let X be a topological space.*

1. *Prove that every map $f\colon \mathbb{R}^n \to X$ is homotopic to a constant map.*
2. *Are $f, g\colon \mathbb{R}^n \to X$ always homotopic? Find a condition on the space X such that all maps from $\mathbb{R}^n$ to X are homotopic.*

Naloga 1.3 *Let $f, g\colon X \to \mathbb{C} - \{0\}$ be such that $|f(x) - g(x)| < |f(x)|$ for all $x \in X$. Prove that f and g are homotopic.*

Naloga 1.4 *Let $n \in \mathbb{N} \cup \{0\}$.*

1. *Are $f, g\colon \mathbb{S}^n \to \mathbb{S}^n$ homotopic?*
2. *Let $f\colon \mathbb{S}^n \to \mathbb{S}^n$ be non surjective map. Prove that f is homotopic to constant map.*
3. *Let $f, g\colon \mathbb{S}^n \to \mathbb{S}^n$ such that $f(x) \neq -g(x)$ for all $x \in \mathbb{S}^n$. Prove that f and g are homotopic.*
4. *Let $f\colon \mathbb{S}^n \to \mathbb{S}^n$ taka, da je $f(x) \neq -x$ za vse $x \in \mathbb{S}^n$. Prove that f is not homotopic to a constant map.*

Naloga 1.5 *Let X be a topological space, $x_0 \in \mathbb{S}^n$ and $f\colon \mathbb{S}^n \to X$. Prove that the following is equivalent:*

1. *The map f is homotopic to a constant map c.*
2. *There exists an extension $F\colon \mathbb{B}^{n+1} \to X$ of the map f.*
3. *$f \simeq c$ (rel x_0).*

Naloga 1.6 *Let $n \in \mathbb{N}$.*

1. *Prove that the inclusion $i\colon \mathbb{S}^{n-1} \hookrightarrow \mathbb{R}^n - \{0^n\}$ is not homotopic to a constant map.*
2. *Let $m \in \mathbb{N}$ and $m > n$. Prove that the inclusion $i\colon \mathbb{S}^{n-1} \hookrightarrow \mathbb{R}^m - \{0^m\}$ is homotopic to a constant map.*

Naloga 1.7 *Let $f, g\colon [-1, 1] \to \mathbb{R}^2 - \{(0,0)\}$ be defined as $f(x) = (x, \sqrt{1-x^2})$ and $g(x) = (x, -\sqrt{1-x^2})$. Prove that $f \simeq g$ and $f \not\simeq g$ (rel $\{-1, 1\}$).*

Naloga 1.8 *Let $N = \{\frac{1}{n} \mid n \in \mathbb{N}\} \cup \{0\}$, $X = \mathbb{I} \times N$, $Y = \{(x, kx) \in \mathbb{R}^2 \mid x \in \mathbb{I}, k \in N\}$ and $A = \mathbb{I} \times \{0\}$. Definirajmo $f(x, y) = (x, 0)$ in $g(x, y) = (x, xy)$.*

1. *Prove that for $f, g\colon X \to Y$ we have $f \simeq g$.*
2. *Prove that for $f, g\colon (X, A) \to (Y, A)$ we have $f \simeq g$.*
3. *Prove that for $f, g\colon X \to Y$ we have $f \simeq g$ (rel A).*